Motion by anisotropic mean curvature as sharp interface limit of an inhomogeneous and anisotropic Allen-Cahn equation *

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Abstract

We consider the spatially inhomogeneous and anisotropic reaction-diffusion equation $u_t = m(x)^{-1} \operatorname{div}[m(x)a_p(x, \nabla u)] + \varepsilon^{-2} f(u)$, involving a small parameter $\varepsilon > 0$ and a bistable nonlinear term whose stable equilibria are 0 and 1. We use a Finsler metric related to the anisotropic diffusion term and work in relative geometry. We prove a weak comparison principle and perform an analysis of both the generation and the motion of interfaces. More precisely, we show that, within the time scale of order $\varepsilon^2 |\ln \varepsilon|$, the unique weak solution u^ε develops a steep transition layer that separates the regions $\{u^\varepsilon \approx 0\}$ and $\{u^\varepsilon \approx 1\}$. Then, on a much slower time scale, the layer starts to propagate. Consequently, as $\varepsilon \to 0$, the solution u^ε converges almost everywhere to 0

*AMS Subject Classifications: 35K55, 35K57, 35B25, 35R35, 58B20.

in Ω_t^- and 1 in Ω_t^+, where Ω_t^- and Ω_t^+ are sub-domains of Ω separated by an interface Γ_t, whose motion is driven by its anisotropic mean curvature. We also prove that the thickness of the transition layer is of order ε.

Key Words: nonlinear PDE, anisotropic diffusion, bistable reaction, inhomogeneity, singular perturbation, Finsler metric, generation of interface, motion by anisotropic mean curvature.

1 Introduction

Evolution laws for interfaces frequently appear in materials science, differential geometry and image processing. In this paper we relate so called diffuse and sharp interface models in which interfaces evolve according to an evolution law, which involves anisotropic and inhomogeneous driving forces. The evolution equations we will consider in particular decrease an inhomogeneous interfacial energy. A diffuse interface model is based on a free energy which includes gradient terms and, in this paper, the energy is assumed to be of the following Ginzburg-Landau type

$$\mathcal{F}(u) = \int_\Omega [a(x, \nabla u) + \frac{1}{\varepsilon^2} W(u)] m(x) dx \,,$$

where $\Omega \subset \mathbb{R}^N$, $N \geq 2$, is a bounded domain with smooth boundary, $\varepsilon > 0$ is a small parameter related to the thickness of a diffuse interfacial layer, W is a double well potential with wells of equal depth and m is a positive function. We will allow a to be x-dependent and anisotropic, i.e. the value of a will depend on the direction of ∇u. Taking the gradient flow of $\mathcal{F}$ with respect to the weighted L^2-inner-product $(u, v) = \int_\Omega u(x)v(x)m(x)dx$ leads to the following initial boundary value problem for an inhomogeneous and anisotropic Allen-Cahn equation

$$(P^\varepsilon) \quad \begin{cases} u_t = \dfrac{1}{m(x)} \operatorname{div} \Big[m(x) a_p(x, \nabla u) \Big] + \dfrac{1}{\varepsilon^2} f(u) & \text{in } \Omega \times (0, +\infty), \\ a_p(x, \nabla u) \cdot \nu = 0 & \text{on } \partial\Omega \times (0, +\infty), \\ u(x, 0) = u_0(x) & \text{in } \Omega, \end{cases}$$

where $f(u) = -W'(u)$, ν is the Euclidean unit normal vector exterior to $\partial\Omega$ and a_p refers to differentiation with respect to the variable corresponding to ∇u. We easily derive that solutions to (P^ε) fulfill

$$\frac{d}{dt} \mathcal{F}(u) = -\int_\Omega (u_t)^2 m(x) dx \leq 0 \,,$$

i.e. $\mathcal{F}$ serves as a Lyapunov function.

It can be shown that under appropriate assumptions, see [10], [22], [23], the energies $\varepsilon\mathcal{F}$ converge in the sense of Γ-convergence to an anisotropic functional defined for hypersurfaces, i.e. in the limit $\varepsilon \to 0$ the interface is sharp. For a smooth hypersurface Γ the limiting energy becomes

$$\int_\Gamma \sqrt{2a(x, n(x))}\, m(x) d\mathcal{H}^{N-1}(x)\,,$$

where n is a suitable Euclidean unit normal vector to Γ and $d\mathcal{H}^{N-1}$ refers to integration with respect to the $(N-1)$-dimensional Hausdorff measure. Anisotropic energies for hypersurfaces can be analyzed in the context of Finsler geometry. If one considers the steepest descent of the anisotropic surface energy in relative geometry, where geometric quantities such as curvature and normal velocity are computed within the context of a Finsler metric, one obtains an anisotropic and inhomogeneous generalization of mean curvature flow. In fact the moving interface Γ_t evolves according to the law

$$(P^0) \quad \begin{cases} \dfrac{m(x)}{\sqrt{2a(x,n)}} V_n = -\operatorname{div}\left[\dfrac{m(x)}{\sqrt{2a(x,n)}}\, a_p(x,n)\right] & \text{on } \Gamma_t, \\ \Gamma_t\Big|_{t=0} = \Gamma_0, \end{cases}$$

where V_n is the normal velocity of Γ_t. We will show below that this equation can be rewritten in the relative geometry associated with a Finsler metric; then it has the form

$$(P^0) \quad \begin{cases} V_{n,\phi} = -\kappa_\phi & \text{on } \Gamma_t, \\ \Gamma_t\Big|_{t=0} = \Gamma_0, \end{cases}$$

where n_ϕ, $V_{n,\phi}$ and κ_ϕ are, respectively, the anisotropic unit normal in the exterior direction, the anisotropic normal velocity of Γ_t in the n_ϕ direction, and the anisotropic mean curvature at each point of Γ_t. In the isotropic homogeneous case one recovers the mean curvature flow $V_n = -\kappa$. We refer to a paper by Bellettini and Paolini [5] and Section 2 for details.

It is the goal of this paper to rigorously prove that Problem (P^ε) converges to the anisotropic inhomogeneous mean curvature flow (P^0), as $\varepsilon \to 0$, and to give an optimal error estimate between the solutions of (P^ε) and those of (P^0). We remark that a formal derivation is already contained in the paper by Bellettini and Paolini [5].

Before going into the details, we note that (P^ε) includes the following equations as special cases: the spatially inhomogeneous diffusion equation

$$u_t = \operatorname{div}(A(x)\nabla u) + \frac{1}{\varepsilon^2} f(u)\,, \tag{1.1}$$

where $A(x)$ is a positive definite symmetric matrix depending on x; the fully anisotropic equation

$$u_t = \operatorname{div}\big(a_p(\nabla u)\big) + \frac{1}{\varepsilon^2} f(u). \tag{1.2}$$

The significant difference between (1.1) and (1.2) is that the anisotropy in (1.2) depends on the solution u itself, while it does not in (1.1). In other words, in (1.2) the dependence of the energy density on the spatial orientation of the interface can be chosen much more general when compared with (1.1) where only ellipsoidal energy densities appear. We refer to Garcke, Nestler and Stoth [15], Barrett, Garcke and Nürnberg [2, 3] for possible anisotropic energy densities. Note also that we allow $a_{pp}(p)$ to be discontinuous at $p = 0$ (see Remark 1.3 below).

We suppose in what follows that $W(u)$ is a double-well potential with equal well-depth, taking its global minimum value at $u = 0$ and $u = 1$. More precisely we assume that $f = -W'$ is smooth and has exactly three zeros $0 < a < 1$ such that

$$f'(0) < 0, \quad f'(a) > 0, \quad f'(1) < 0, \tag{1.3}$$

and that

$$\int_0^1 f(u)du = 0. \tag{1.4}$$

Remark 1.1. Note that we could also consider the case where f is slightly unbalanced by order ε so that $\int_0^1 f(u)du = O(\varepsilon)$ stands instead of (1.4). In this case, the singular limit of (P^ε) will have an additional driving force term in (P^0). See Remark 3.2 for details. □

The assumptions concerning the anisotropic term are the following.

(i) $a(x,p)$ is a real valued function, of class $C^{3+\vartheta}_{loc}$ (for some $0 < \vartheta < 1$) on $\overline{\Omega} \times \mathbb{R}^N \backslash \{0\}$;

(ii) $a(x,p)$ is positive on $\overline{\Omega} \times \mathbb{R}^N \backslash \{0\}$;

(iii) $a(x,\cdot)$ is strictly convex for all $x \in \overline{\Omega}$;

(iv) $a(x,p)$ is homogeneous of degree two in the p variable, i.e.

$$a(x,\alpha p) = \alpha^2 a(x,p) \quad \text{for all } (x,p) \in \overline{\Omega} \times \mathbb{R}^N \backslash \{0\}, \text{ all } \alpha \neq 0. \tag{1.5}$$

If p is given by $p = (p_1, \cdots, p_N)$, the vector valued function a_p is defined by $a_p(x,p) = \Big(\frac{\partial a}{\partial p_1}(x,p), \cdots, \frac{\partial a}{\partial p_N}(x,p)\Big)$, and the matrix valued function a_{pp} by $a_{pp}(x,p) = \Big(\frac{\partial^2 a}{\partial p_j \partial p_i}(x,p)\Big)$. Moreover, for a vector $p = (p_1, \cdots, p_N)$ and a matrix $A = (a_{ij})$, we use the notations

$$|p| = \max_i |p_i| \qquad \text{and} \qquad |A| = \max_{i,j} |a_{ij}|.$$

Remark 1.2. The fact that a is homogeneous of degree two implies that, for all $(x,p) \in \overline{\Omega} \times \mathbb{R}^N \backslash \{0\}$, all $\alpha \neq 0$,

$$\begin{aligned} a_p(x,\alpha p) = \alpha a_p(x,p),\ a_{pp}(x,\alpha p) = a_{pp}(x,p), \\ a_p(x,\alpha p)\cdot p = 2\alpha a(x,p), \\ a_{pp}(x,\alpha p)p = a_p(x,p). \end{aligned}$$

By setting $a(x,0)=0$ and $a_p(x,0)=0$, one can understand that $a(x,p)$ is of class C^1 on the whole of $\overline{\Omega}\times\mathbb{R}^N$. $\square$

Remark 1.3. In many important applications in physics, $a_{pp}(x,p)$ is discontinuous at $p=0$ and this makes Problem (P^ε) singularly parabolic. Because of lack of uniform parabolicity, our analysis becomes more involved than the case (1.1) or the case of isotropic Allen-Cahn equation studied in [1], [11], [12], [19, 20]. $\square$

We assume that $m:\Omega\to(0,+\infty)$ is a function of class C^2 such that $0<m_1\le m(x)\le m_2<+\infty$ for any $x\in\Omega$, and that ∇m and $D^2 m$ are in $L^\infty(\Omega)$, where $D^2m(x) := \Big(\dfrac{\partial^2 m}{\partial x_j \partial x_i}(x)\Big)$.

Remark 1.4. Observe that

$$\frac{1}{m(x)}\,\mathrm{div}\Big[m(x)a_p(x,\nabla u)\Big] = \mathrm{div}\,a_p(x,\nabla u) + \nabla \log m(x)\cdot a_p(x,\nabla u)\,, \quad (1.6)$$

so that our equation contains the generalized Allen-Cahn equations discussed in Bellettini, Paolini [5], Bellettini, Paolini and Venturini [6]. $\square$

We also assume that the initial data $u_0\in C^2(\overline{\Omega})$, and define C_0 as

$$C_0 := \|u_0\|_{C^0(\overline{\Omega})} + \|\nabla u_0\|_{C^0(\overline{\Omega})} + \|D^2u_0\|_{C^0(\overline{\Omega})}. \quad (1.7)$$

Furthermore we define the "initial interface" Γ_0 by

$$\Gamma_0 := \{x\in\Omega,\ u_0(x)=a\},$$

and suppose that Γ_0 is a $C^{3+\vartheta}$ closed hypersurface without boundary ($0<\vartheta<1$), such that, n being the Euclidian unit normal vector exterior to Γ_0,

$$\Gamma_0 \subset\subset \Omega \quad \text{and} \quad \nabla u_0(x)\neq 0 \quad \text{if } x\in\Gamma_0, \quad (1.8)$$

$$u_0 > a \quad \text{in} \quad \Omega_0^+, \quad u_0<a \quad \text{in} \quad \Omega_0^-, \quad (1.9)$$

where Ω_0^- denotes the region enclosed by Γ_0 and Ω_0^+ the region enclosed between $\partial\Omega$ and Γ_0.

For $T>0$, we set $Q_T=\Omega\times(0,T)$. We define below a notion of weak solutions of Problem (P^ε). For this definition, it is sufficient to only suppose that $u_0\in H^1(\Omega)\cap L^\infty(\Omega)$.

Definition 1.5. A function $u^\varepsilon \in L^2(0,T;H^1(\Omega)) \cap L^\infty(Q_T)$ is a weak solution of Problem (P^ε), if

(i) $u_t^\varepsilon \in L^2(Q_T)$,

(ii) $a_p(x, \nabla u^\varepsilon(x,t)) \in L^\infty(0,T;L^2(\Omega))$,

(iii) $u^\varepsilon(x,0) = u_0(x)$ for almost all $x \in \Omega$,

(iv) u^ε satisfies the integral equality

$$\int_0^t \int_\Omega \Big[u_t^\varepsilon \varphi + a_p(x, \nabla u^\varepsilon) \cdot \nabla \varphi - \frac{1}{\varepsilon^2} f(u^\varepsilon) \varphi \Big] m(x) dx dt = 0 \,, \qquad (1.10)$$

for all nonnegative function $\varphi \in L^2(0,T;H^1(\Omega)) \cap L^\infty(Q_T)$ and for all $t \in (0,T)$.

One may prove, using monotonicity and compactness arguments as is done in [7], [9], that Problem (P^ε) possesses a unique weak solution which we denote by u^ε. As $\varepsilon \to 0$, the qualitative behavior of this solution is the following. In the very early stage, the anisotropic diffusion term is negligible compared with the reaction term $\varepsilon^{-2} f(u)$. Hence, rescaling time by $\tau = t/\varepsilon^2$, the equation is well approximated by the ordinary differential equation $u_\tau = f(u)$. In view of the bistable nature of f, u^ε quickly approaches the values 0 or 1, the stable equilibria of the ODE, and an interface is formed between the regions $\{u^\varepsilon \approx 0\}$ and $\{u^\varepsilon \approx 1\}$. Once such an interface has been developed, the anisotropic diffusion term becomes large near the interface, and comes to balance with the reaction term so that the interface starts to propagate, on a much slower time scale.

To understand such interfacial behavior, we have to study the singular limit of (P^ε) as $\varepsilon \to 0$. Then the limit solution $\tilde{u}(x,t)$ is a step function taking the values 0 and 1 on the sides of the moving interface Γ_t. In the case of the usual Allen-Cahn equation, it is well known that Γ_t evolves according to the mean curvature flow $V_n = -\kappa$ and we will show in this paper that the sharp interface limit of (P^ε) is given by (P^0).

Using the theory of analytic semigroups (see e.g. Lunardi [18]) it is possible to show that the limit Problem (P^0) possesses locally in time a unique smooth solution. More precisely, there exists a $T > 0$ such that Problem (P^0) has a unique solution $\Gamma = \bigcup_{0 \le t < T} (\Gamma_t \times \{t\})$ which satisfies $\Gamma \in C^{3+\vartheta,(3+\vartheta)/2}$. For proofs of the local in time existence of solutions of related limit problems, we also refer the reader to [17] and the discussion at the end of Chapter 1 in [16].

For each $t \in (0,T)$, we define Ω_t^- as the region enclosed by the hypersurface Γ_t and Ω_t^+ as the region lying between $\partial\Omega$ and Γ_t. Furthermore we

define a step function $\tilde{u}(x,t)$ by

$$\tilde{u}(x,t) = \begin{cases} 1 & \text{in } \Omega_t^+ \\ 0 & \text{in } \Omega_t^- \end{cases} \quad \text{for } t \in (0,T). \tag{1.11}$$

It is convenient to present our main result, Theorem 1.6, in the form of a convergence theorem, mixing generation and propagation. It describes the profile of the solution after a very short initial period. It asserts that: given a virtually arbitrary initial data u_0, the solution u^ε quickly becomes close to 0 or 1, except in a small neighborhood of the initial interface Γ_0, creating a steep transition layer around Γ_0 (*generation of interface*). The time t^ε for the generation of interface is of order $\varepsilon^2|\ln\varepsilon|$. The theorem then states that the solution u^ε remains close to the step function $\tilde{u}$ on the time interval (t^ε, T) (*motion of interface*). Moreover, as is clear from the estimates in the theorem, the thickness of the transition layer is of order ε.

Theorem 1.6 (Generation, motion and thickness of interface)**.** *Let η be an arbitrary constant satisfying $0 < \eta < \min(a, 1-a)$ and set*

$$\mu = f'(a).$$

Then there exist positive constants ε_0 and C such that, for all $\varepsilon \in (0, \varepsilon_0)$ and for almost all (x,t) such that $t^\varepsilon \le t \le T$, where

$$t^\varepsilon := \mu^{-1}\varepsilon^2|\ln\varepsilon|\,, \tag{1.12}$$

we have,

$$u^\varepsilon(x,t) \in \begin{cases} [-\eta, 1+\eta] & \textit{if} \quad x \in \mathcal{N}_{C\varepsilon}(\Gamma_t) \\ [-\eta, \eta] & \textit{if} \quad x \in \Omega_t^- \setminus \mathcal{N}_{C\varepsilon}(\Gamma_t) \\ [1-\eta, 1+\eta] & \textit{if} \quad x \in \Omega_t^+ \setminus \mathcal{N}_{C\varepsilon}(\Gamma_t), \end{cases} \tag{1.13}$$

where $\mathcal{N}_r(\Gamma_t) := \{x \in \Omega, dist_\phi(x,\Gamma_t) < r\}$ denotes the r-neighborhood of Γ_t; by $dist_\phi(x,\Gamma_t)$, we mean the δ_ϕ distance to the set Γ_t, where δ_ϕ is the distance associated to a Finsler metric, whose definition is to be given in Section 2.

Corollary 1.7 (Convergence)**.** *As $\varepsilon \to 0$, the solution u^ε converges to $\tilde{u}$ almost everywhere in $\bigcup_{0<t<T}(\Omega_t^\pm \times \{t\})$.*

The organization of this paper is as follows. In Section 2, we recall notations and results concerning Finsler metrics that give a natural and efficient framework for dealing with anisotropic problems. In Section 3, we perform formal asymptotic expansions in order to derive the equation for the motion

of interface, and collect useful estimates on stationary solutions of related problems. In Section 4 we prove a weak comparison principle for Problem (P^ε). Such a comparison principle is rather standard, but, in view of the fact that $a_{pp}(x,p)$ does not exist at $p=0$, we give a complete proof for the convenience of the reader. In Section 5, we prove results on the generation of interface. For the study of this early time range we construct sub- and super-solutions by modifying the solution of the ordinary differential equation $u_t = \varepsilon^{-2} f(u)$. In Section 6, we construct another pair of sub- and super-solutions by using the first two terms of the formal asymptotic expansion given in Section 3. They are used to study the motion of interface in the later stage. In Section 7, by fitting these two pairs of sub- and super-solutions together, we prove our main results for (P^ε): Theorem 1.6 and its corollary.

Let us mention some earlier works on anisotropic problems related to (P^ε). In [4], Bellettini, Colli Franzone and Paolini study a problem that is slightly more general than (P^ε) — by allowing f to be unbalanced in the same way as in Remarks 1.1, 3.2 of the present paper — and derive a very fine error estimate between the formal asymptotic and actual solutions of (P^ε). We also refer to the articles [13, 14], by Elliott and Schätzle, on a similar but slightly different problem where the potential $W(u)$ is a double obstacle type, namely $W(u) = +\infty$ for $u \notin (0,1)$. For the spatially homogeneous case $a(x,p) = a(p)$ they prove convergence of the anisotropic diffusion problem to an anisotropic curvature flow similar to (P^0). Note that their second paper [14] considers a kinetic term of the form $\beta(\nabla u)u_t$, which makes the meaning of solutions very weak, hence they are treated in the framework of viscosity solutions.

However, in these papers, the authors consider only a very restricted class of initial data, namely those having a specific profile with well-developed transition layer. More precisely they prove that if the initial data is very close to the typical profile that appears in the formal asymptotic expansions of the moving interface, then the solution remains close to the formal asymptotic for $0 \le t \le T$. In other words the generation of interface from arbitrary initial data is not studied there. Summarizing, they have obtained a very fine error estimate — of order $O(\varepsilon^2)$ or higher — between the solutions of specific initial data and formal asymptotic, while, in the present paper, we consider convergence of solutions of (P^ε) with virtually arbitrary initial data to solutions of (P^0), with an error estimate of order $O(\varepsilon)$. Therefore, the two results are both for the convergence of (P^ε) to (P^0), but they are of different nature. Note that, as far as the thickness of the interface is concerned, our $O(\varepsilon)$ estimate is optimal (see [1] for details).

In [8], Beneš, Hilhorst and Weidenfeld study both the generation and the motion of interface for an anisotropic Allen-Cahn equation which is related to ours. Nevertheless, their equation is slightly less general since they do not allow x-dependence in $a(x,p)$. Moreover, with their sub- and super-

solutions, they cannot achieve the optimal $O(\varepsilon)$ estimate of the thickness of the interface.

For numerical simulations for problems (P^ε) and (P^0) we refer to Beneš, Mikula [9], Garcke, Nestler, Stoth [15], Barrett, Garcke, Nürnberg [2, 3] and Paolini [24].

2 Finsler metrics and the anisotropic context

In this section we explain the technique of Bellettini, Paolini [5], Bellettini, Paolini and Venturini [6] to apply Finsler metric to analyze anisotropic non-linear problems. The idea is to endow $\mathbb{R}^N$ with the distance obtained by integrating the Finsler metric which makes otherwise lengthy computations remarkably simpler. For the convenience of the reader, we first recall basic properties of Finsler metrics as stated in [5], [6]. For more details and proofs, see these references.

2.1 Finsler metrics

Suppose that $\phi : \Omega \times \mathbb{R}^N \to [0, +\infty)$ is a continuous function satisfying the properties

$$\phi(x, \alpha\xi) = |\alpha|\phi(x, \xi) \quad \text{for all } (x, \xi) \in \Omega \times \mathbb{R}^N \quad \text{and all } \alpha \in \mathbb{R}, \tag{2.1}$$

$$\lambda_0|\xi| \le \phi(x, \xi) \le \Lambda_0|\xi| \quad \text{for all } (x, \xi) \in \Omega \times \mathbb{R}^N, \tag{2.2}$$

for two suitable constants $0 < \lambda_0 \le \Lambda_0 < +\infty$. We say that ϕ is strictly convex if, for any $x \in \Omega$, the map $\xi \mapsto \phi^2(x, \xi)$ is strictly convex on $\mathbb{R}^N$. We shall indicate by

$$B_\phi(x) = \{\xi \in \mathbb{R}^N, \phi(x, \xi) \le 1\}$$

the unit sphere of ϕ at $x \in \Omega$.

The dual function $\phi^0 : \Omega \times \mathbb{R}^N \to [0, +\infty)$ of ϕ is defined by

$$\phi^0(x, \xi^*) = \sup\left\{\xi^* \cdot \xi,\ \xi \in B_\phi(x)\right\}, \tag{2.3}$$

for any $(x, \xi) \in \Omega \times \mathbb{R}^N$. One can prove that ϕ^0 is continuous, convex, satisfies properties (2.1) and (2.2), and that ϕ^{00}, the dual function of ϕ^0, coincides with the convex envelope of ϕ with respect to ξ.

We say that ϕ is a (strictly convex smooth) Finsler metric, and we shall write $\phi \in \mathcal{M}(\Omega)$ if, in addition to properties (2.1) and (2.2), ϕ and ϕ^0 are strictly convex and of class C^2 on $\Omega \times \mathbb{R}^N \setminus \{0\}$. In particular $\phi^{00} = \phi$.

We denote by δ_ϕ the integrated distance associated to $\phi \in \mathcal{M}(\Omega)$, that is, for any $(x, y) \in \Omega$, we set

$$\delta_\phi(x, y) = \inf\Big\{\int_0^1 \phi(\gamma(t), \dot\gamma(t))dt\,;\ \gamma \in W^{1,1}\big([0,1]; \Omega\big), \gamma(0) = x, \gamma(1) = y\Big\}. \tag{2.4}$$

In the special case of the Euclidian metric, the function ϕ is given by $\phi(x,p) = \phi(p) = ({p_1}^2 + \cdots + {p_N}^2)^{1/2}$, so that δ_ϕ reduces to the usual distance.

Given $\phi \in \mathcal{M}(\Omega)$ and $x \in \Omega$, let $T^0(x,\cdot) : \mathbb{R}^N \to \mathbb{R}^N$ be the map defined by

$$T^0(x,\xi^*) = \begin{cases} \phi^0(x,\xi^*)\phi^0_p(x,\xi^*) & \text{if } \xi^* \in \mathbb{R}^N \setminus \{0\} \\ 0 & \text{if } \xi^* = 0. \end{cases} \tag{2.5}$$

Here ϕ^0_p denotes the gradient with respect to p whenever we regard $\phi^0(x,p)$ as a function of two variables x and p.

If $u : \Omega \to \mathbb{R}$ is a smooth function with non-vanishing gradient, we define the anisotropic gradient by

$$\nabla_\phi u = T^0(x,\nabla u) = \phi^0(x,\nabla u)\phi^0_p(x,\nabla u). \tag{2.6}$$

If $\eta : \Omega \to \mathbb{R}^N$ is a smooth vector field, we define the m-divergence operator by

$$\operatorname{div}_m \eta = \frac{1}{m(x)} \operatorname{div} \big[m(x)\eta\big] = \operatorname{div} \eta + \nabla \log m(x) \cdot \eta\,, \tag{2.7}$$

and then the m-anisotropic Laplacian by

$$\Delta_{\phi,m} u = \operatorname{div}_m \nabla_\phi u. \tag{2.8}$$

Note that in [5], [6] m is related to ϕ while in the present paper m is a given function independent of ϕ. Nonetheless, in the sequel, we shall use the simpler notation $\Delta_\phi := \Delta_{\phi,m}$.

As in the isotropic case, if Γ_t is a smooth hypersurface of Ω at time t, and n the outer normal vector to Γ_t (in the Euclidian sense), we define n_ϕ the ϕ-normal vector to Γ_t and κ_ϕ the ϕ-mean curvature of Γ_t by

$$n_\phi = \phi^0_p(x,n), \quad \kappa_\phi = \operatorname{div}_m n_\phi. \tag{2.9}$$

Furthermore, if ψ is a smooth function with non-vanishing gradient such that $\Gamma_t = \{x \in \Omega,\ \psi(x,t) = 0\}$, and ψ is positive outside Γ_t and negative inside, then

$$n = \frac{\nabla\psi}{|\nabla\psi|}, \qquad n_\phi = \phi^0_p(x,\nabla\psi), \tag{2.10}$$

$$\kappa = \operatorname{div} \frac{\nabla\psi}{|\nabla\psi|}, \qquad \kappa_\phi = \operatorname{div}_m \phi^0_p(x,\nabla\psi), \tag{2.11}$$

on Γ_t. We also define the normal velocity of Γ_t and the ϕ-normal velocity of Γ_t by

$$V_n = -\frac{\psi_t}{|\nabla\psi|}, \qquad V_{n,\phi} = -\frac{\psi_t}{\phi^0(x,\nabla\psi)}. \tag{2.12}$$

To conclude these preliminaries, we quote a theorem proved in [6].

Theorem 2.1. *Let Ω be connected, and let $\phi \in \mathcal{M}(\Omega)$. Let δ_ϕ be the integrated distance associated to ϕ. Let $C \subseteq \Omega$ be a closed set, and let $dist_\phi(x, C)$ be the δ_ϕ distance to the set C defined by*

$$dist_\phi(x, C) = \inf \big\{ \delta_\phi(x, y) \ , \ y \in C \big\}. \tag{2.13}$$

Then

$$\phi^0\big(x, \nabla dist_\phi(x, C)\big) = 1 \, , \tag{2.14}$$

at each point $x \in \Omega \setminus C$ where $dist_\phi(\cdot, C)$ is differentiable.

In the special case of the Euclidian metric, note that (2.14) reduces to the property that $|\nabla d| = 1$.

2.2 Application to the anisotropic Allen-Cahn equation

We set, for all $(x, p) \in \Omega \times \mathbb{R}^N$,

$$\phi^0(x, p) = \sqrt{2a(x, p)}. \tag{2.15}$$

First, since $a(x, \cdot)$ is homogeneous of degree two, ϕ^0 satisfies assumptions (2.1) and (2.2) with the constants

$$\lambda_0 = [2 \min_{x \in \overline{\Omega}, |p|=1} a(x, p)]^{1/2} > 0 \quad \text{and} \quad \Lambda_0 = [2 \max_{x \in \overline{\Omega}, |p|=1} a(x, p)]^{1/2} > 0. \tag{2.16}$$

By the hypotheses on $a(x, p)$, we see that ϕ^0 is strictly convex and of class C^2 on $\Omega \times \mathbb{R}^N \setminus \{0\}$; moreover, by Remark 1.2, ϕ^0 is continuous on the whole of $\Omega \times \mathbb{R}^N$. It follows that ϕ is a Finsler metric and the above theory applies. We have

$$T^0(x, p) = \begin{cases} a_p(x, p) & \text{if } p \in \mathbb{R}^N \setminus \{0\} \\ 0 & \text{if } p = 0. \end{cases} \tag{2.17}$$

Let $\Gamma = \bigcup_{0 \le t < T} (\Gamma_t \times \{t\})$ be the unique solution of the limit geometric motion Problem (P^0) and let $\widetilde{d}$ be the signed distance function to Γ defined by

$$\widetilde{d}(x, t) = \begin{cases} \text{dist}(x, \Gamma_t) & \text{for } x \in \Omega_t^+, \\ -\text{dist}(x, \Gamma_t) & \text{for } x \in \Omega_t^-, \end{cases} \tag{2.18}$$

where $\text{dist}(x, \Gamma_t)$ is the distance from x to the hypersurface Γ_t in Ω. Let $\widetilde{d}_\phi$ be the anisotropic signed distance function to Γ defined by

$$\widetilde{d}_\phi(x, t) = \begin{cases} \text{dist}_\phi(x, \Gamma_t) & \text{for } x \in \Omega_t^+, \\ -\text{dist}_\phi(x, \Gamma_t) & \text{for } x \in \Omega_t^-, \end{cases} \tag{2.19}$$

where $\text{dist}_\phi(x, \Gamma_t)$ denotes the δ_ϕ distance to the set Γ_t defined in (2.13). By Theorem 2.1, the following equality holds

$$2a(x, \nabla \widetilde{d}_\phi(x, t)) = 1 \qquad \text{in a neighborhood of } \Gamma_t. \tag{2.20}$$

By setting $\psi = \widetilde{d}$ and $\psi = \widetilde{d}_\phi$ in the second equalities in (2.10), (2.11), (2.12), we obtain two equivalent expressions of the ϕ-normal vector, the ϕ-mean curvature and the ϕ-normal velocity:

$$n_\phi = \frac{1}{\sqrt{2a(x, \nabla \widetilde{d})}}\, a_p(x, \nabla \widetilde{d}) = \ a_p(x, \nabla \widetilde{d}_\phi), \tag{2.21}$$

$$\kappa_\phi = \operatorname{div}_m \left[\frac{1}{\sqrt{2a(x, \nabla \widetilde{d})}}\, a_p(x, \nabla \widetilde{d}) \right] = \operatorname{div}_m \left[a_p(x, \nabla \widetilde{d}_\phi) \right], \tag{2.22}$$

$$V_{n,\phi} = - \frac{1}{\sqrt{2a(x, \nabla \widetilde{d})}}\, \widetilde{d}_t = -(\widetilde{d}_\phi)_t. \tag{2.23}$$

The end of this section is devoted to the anisotropic Laplacian

$$\Delta_\phi u = \frac{1}{m(x)} \operatorname{div} \left[m(x) a_p(x, \nabla u) \right] \tag{2.24}$$

$$= \operatorname{div} a_p(x, \nabla u) + \nabla \log m(x) \cdot a_p(x, \nabla u). \tag{2.25}$$

In the case of Finsler metrics, it turns out that the term $\Delta_\phi u$ may be less regular than Δu. Nevertheless, we show below a boundedness property of the anisotropic Laplacian (see [8] for a related property).

Lemma 2.2. *There exists a positive constant C_L such that, for all $u \in C^{2,1}(Q_T)$, the following inequality holds:*

$$|\Delta_\phi u(x,t)| \le C_L (|\nabla u(x,t)| + |D^2 u(x,t)|) \qquad \textit{for all } (x,t) \in Q_T. \tag{2.26}$$

Proof. In view of (2.25), it is sufficient to deal with the term $\operatorname{div} a_p(x, \nabla u)$. We can, with no loss of generality, ignore the dependence on time.

First, assume that x is such that $\nabla u(x) \neq 0$. Regarding $a(x,p)$ as a function of two variables x and $p = (p_1, \cdots, p_n)$, we obtain, by a straightforward calculation, that

$$\operatorname{div} a_p(x, \nabla u(x)) = \sum_j \frac{\partial^2 a}{\partial x_j \partial p_j}(x, \nabla u(x)) + \sum_{i,j} \frac{\partial^2 a}{\partial p_i \partial p_j}(x, \nabla u(x)) \frac{\partial^2 u}{\partial x_i \partial x_j}(x). \tag{2.27}$$

It follows from the homogeneity properties that

$$\begin{aligned} |\operatorname{div} a_p(x, \nabla u(x))| \le & |\nabla u(x)| \sum_j \max_{y \in \overline{\Omega}, |p|=1} \left| \frac{\partial^2 a}{\partial x_j \partial p_j}(y,p) \right| \\ & + |D^2 u(x)| \sum_{i,j} \max_{y \in \overline{\Omega}, |p|=1} \left| \frac{\partial^2 a}{\partial p_i \partial p_j}(y,p) \right|, \end{aligned}$$

where we have used that a is of class C^2 on the compact set $\overline{\Omega} \times \{|p| = 1\}$. This proves (2.26) under the assumption $\nabla u(x) \neq 0$.

Now assume that x is such that $\nabla u(x) = 0$. We have to proceed in a slightly different way since $a_{pp}(x, 0)$ does not make sense. The operator $a_p(x, \cdot)$ is homogeneous of degree one so that, for any direction ζ,

$$t^{-1}(a_p(x, t\zeta) - a_p(x, 0)) = a_p(x, \zeta).$$

We denote by $(e_1, \cdots, e_N)$ the Euclidian basis of $\mathbb{R}^N$. It follows from the above equality that $a_p(x, \cdot)$ admits at the point 0 partial derivatives in any direction e_i and

$$\frac{\partial a_p(x, \cdot)}{\partial p_i}(0) = a_p(x, e_i)\,, \tag{2.28}$$

which, in turn, implies that

$$\frac{\partial}{\partial p_i}\frac{\partial a}{\partial p_j}(x, 0) = \frac{\partial a}{\partial p_j}(x, e_i). \tag{2.29}$$

Note that, since $a_p(x, \cdot)$ is homogeneous of degree one, the first term in (2.27) vanishes at the point $(x, 0)$. It follows from (2.27) and (2.29) that, in the case where $\nabla u(x) = 0$,

$$\begin{aligned} |\operatorname{div} a_p(x, \nabla u(x))| &= \Big|\sum_{i,j} \frac{\partial a}{\partial p_j}(x, e_i)\frac{\partial^2 u}{\partial x_i \partial x_j}(x)\Big| \\ &\leq |D^2 u(x)| \sum_{i,j} \max_{y \in \overline{\Omega}, |p|=1} \Big|\frac{\partial a}{\partial p_j}(y, p)\Big|\,, \end{aligned}$$

which proves (2.26) in this case as well. $\square$

3 Formal derivation of the interface motion equation

In this section we derive the equation of interface motion corresponding to Problem (P^ε) by using a formal asymptotic expansion. The resulting interface equation can be regarded as the singular limit of (P^ε) as $\varepsilon \to 0$. Our argument goes basically along the same lines with the formal derivation given by Nakamura, Matano, Hilhorst and Schätzle [21]: the first two terms of the asymptotic expansion determine the interface equation. Though our analysis in this section is for the most part formal, the results we obtain will help the rigorous analysis in later sections.

Let u^ε be the solution of (P^ε). Let $\Gamma = \bigcup_{0 \leq t < T} \Gamma_t \times \{t\}$ be the solution of the limit geometric motion problem and $\widetilde{d_\phi}$ the anisotropic signed distance

function to Γ defined in (2.19). We then define

$$Q_T^+ = \bigcup_{0<t<T} \Omega_t^+ \times \{t\}, \qquad Q_T^- = \bigcup_{0<t<T} \Omega_t^- \times \{t\}.$$

We also assume that the solution u^ε has — one the one hand — the outer expansions (away from the interface Γ),

$$u^\varepsilon(x,t) = \tilde{u}(x,t) + \varepsilon u_1^\pm(x,t) + \varepsilon^2 u_2^\pm(x,t) + \cdots \quad \text{in } \; Q_T^\pm, \tag{3.1}$$

where $\tilde{u}$ is the step function defined in (1.11), and — on the other hand — the inner expansion (near Γ)

$$u^\varepsilon(x,t) = U_0(x,t,\xi) + \varepsilon U_1(x,t,\xi) + \varepsilon^2 U_2(x,t,\xi) + \cdots, \tag{3.2}$$

near Γ (the inner expansion), where $U_j(x,t,z)$, $j = 0,1,2,\cdots$, are defined for $x \in \overline{\Omega}$, $t \geq 0$, $z \in \mathbb{R}$. The stretched space variable $\xi := \widetilde{d}_\phi(x,t)/\varepsilon$ gives exactly the right spatial scaling to describe the sharp transition between the regions $\{u^\varepsilon \approx 0\}$ and $\{u^\varepsilon \approx 1\}$. We normalize U_0 in such a way that

$$U_0(x,t,0) = a$$

(normalization conditions). To make the inner and outer expansions consistent, we require that

$$\begin{aligned} U_0(x,t,+\infty) &= 1, \qquad U_k(x,t,+\infty) = u_k^+(x,t),\\ U_0(x,t,-\infty) &= 0, \qquad U_k(x,t,-\infty) = u_k^-(x,t), \end{aligned} \tag{3.3}$$

for all $k \geq 1$ (matching conditions).

In what follows we will substitute the inner expansion (3.2) into the parabolic equation in Problem (P^ε) and collect the ε^{-2} and ε^{-1} terms. For this purpose, note that if $V = V(x,t,z)$ and $v(x,t) = V(x,t,\xi)$ are real valued functions then $\nabla v = \varepsilon^{-1} V_z \nabla \widetilde{d}_\phi + \nabla_x V$ and $v_t = \varepsilon^{-1}(\widetilde{d}_\phi)_t V_z + V_t$; if v and V are vector valued functions we obtain $\operatorname{div} v = \varepsilon^{-1} \nabla \widetilde{d}_\phi \cdot V_z + \operatorname{div}_x V$. In the following, we shall use the properties stated in Remark 1.2. A straightforward computation yields

$$\begin{aligned} u_t^\varepsilon &= \frac{1}{\varepsilon}(\widetilde{d}_\phi)_t U_{0z} + U_{0t} + (\widetilde{d}_\phi)_t U_{1z} + \varepsilon U_{1t} + \cdots \\ \nabla u^\varepsilon &= \frac{1}{\varepsilon} U_{0z} \nabla \widetilde{d}_\phi + \nabla_x U_0 + U_{1z} \nabla \widetilde{d}_\phi + \varepsilon \nabla_x U_1 + \cdots \\ a_p(x, \nabla u^\varepsilon) &= \frac{1}{\varepsilon} a_p(x, U_{0z} \nabla \widetilde{d}_\phi + \varepsilon \nabla_x U_0 + \varepsilon U_{1z} \nabla \widetilde{d}_\phi + \varepsilon^2 \nabla_x U_1 + \cdots) \\ &= \frac{1}{\varepsilon} a_p(x, U_{0z} \nabla \widetilde{d}_\phi) + a_{pp}(x, U_{0z} \nabla \widetilde{d}_\phi)(\nabla_x U_0 + U_{1z} \nabla \widetilde{d}_\phi) + \cdots \\ &= \frac{1}{\varepsilon} U_{0z} a_p(x, \nabla \widetilde{d}_\phi) + a_{pp}(x, \nabla \widetilde{d}_\phi)(\nabla_x U_0 + U_{1z} \nabla \widetilde{d}_\phi) + \cdots. \end{aligned}$$

It follows that

$$\nabla \log m(x) \cdot a_p(x, \nabla u^\varepsilon) = \frac{1}{\varepsilon} U_{0z} \nabla \log m(x) \cdot a_p(x, \nabla \widetilde{d}_\phi) + \cdots$$

and that

$$\begin{aligned}
\operatorname{div} a_p(x, \nabla u^\varepsilon) =& \frac{1}{\varepsilon} \nabla \widetilde{d}_\phi \cdot \partial_z (a_p(x, \nabla u^\varepsilon)) + \operatorname{div}_x (a_p(x, \nabla u^\varepsilon)) \\
=& \frac{\nabla \widetilde{d}_\phi}{\varepsilon} \cdot \Big[\frac{U_{0zz}}{\varepsilon} a_p(x, \nabla \widetilde{d}_\phi) + a_{pp}(x, \nabla \widetilde{d}_\phi)(\nabla_x U_{0z} + U_{1zz} \nabla \widetilde{d}_\phi) \Big] \\
& + \frac{1}{\varepsilon} \big[\nabla_x U_{0z} \cdot a_p(x, \nabla \widetilde{d}_\phi) + U_{0z} \operatorname{div} a_p(x, \nabla \widetilde{d}_\phi) \big] + \cdots \\
=& \frac{1}{\varepsilon^2} U_{0zz} 2a(x, \nabla \widetilde{d}_\phi) + \frac{1}{\varepsilon} \Big[2a(x, \nabla \widetilde{d}_\phi) U_{1zz} \\
& + 2 \nabla_x U_{0z} \cdot a_p(x, \nabla \widetilde{d}_\phi) + U_{0z} \operatorname{div} a_p(x, \nabla \widetilde{d}_\phi) \Big] + \cdots ,
\end{aligned}$$

where the functions U_i $(i = 0, 1)$, as well as their derivatives, are taken at the point $(x, t, \widetilde{d}_\phi(x, t)/\varepsilon)$. Hence, in view of (2.20), we obtain

$$\begin{aligned}
\operatorname{div} a_p(x, \nabla u^\varepsilon) =& \frac{1}{\varepsilon^2} U_{0zz} + \frac{1}{\varepsilon} [U_{1zz} + 2 \nabla_x U_{0z} \cdot a_p(x, \nabla \widetilde{d}_\phi) \\
& + U_{0z} \operatorname{div} a_p(x, \nabla \widetilde{d}_\phi)] + \cdots .
\end{aligned}$$

We also use the expansion

$$f(u^\varepsilon) = f(U_0) + \varepsilon U_1 f'(U_0) + \cdots .$$

Next, we substitute the above expressions in the partial differential equation in Problem (P^ε). Collecting the ε^{-2} terms yields

$$U_{0zz} + f(U_0) = 0. \tag{3.4}$$

In view of the normalization and matching conditions, we can now assert that $U_0(x, t, z) = U_0(z)$, where U_0 is the unique solution of the one-dimensional stationary problem

$$\begin{cases} {U_0}'' + f(U_0) = 0, \\ U_0(-\infty) = 0, \quad U_0(0) = a, \quad U_0(+\infty) = 1. \end{cases} \tag{3.5}$$

This solution represents the first approximation of the profile of a transition layer around the interface observed in the stretched coordinates. We recall standard estimates on U_0.

Lemma 3.1. *There exist positive constants C and λ such that*

$$\begin{aligned}
0 < 1 - U_0(z) \le C e^{-\lambda |z|} \quad & \textit{for } z \ge 0, \\
0 < U_0(z) \le C e^{-\lambda |z|} \quad & \textit{for } z \le 0.
\end{aligned}$$

In addition to this ${U_0}' > 0$ and, for all $j = 1, 2$,

$$|D^j U_0(z)| \le C e^{-\lambda |z|} \quad \textit{for } z \in \mathbb{R}.$$

Since U_0 depends only on the variable z, we have $\nabla_x {U_0}' = 0$. Then, by collecting the ε^{-1} terms, we obtain

$$U_{1zz} + f'(U_0)U_1 = (\widetilde{d}_\phi)_t {U_0}' - \Delta_\phi \widetilde{d}_\phi \, {U_0}' \,, \tag{3.6}$$

which can be seen as a linearized problem for (3.4). The solvability condition for the above equation, which is a variant of the Fredholm alternative, plays the key role in deriving the equation of interface motion. It is given by

$$\int_{\mathbb{R}} \Big[(\widetilde{d}_\phi)_t(x,t) - \Delta_\phi \widetilde{d}_\phi(x,t) \Big] {U_0'}^2(z) dz = 0 \,,$$

for all $(x,t) \in Q_T$. It follows that $(\widetilde{d}_\phi)_t = \Delta_\phi \widetilde{d}_\phi$. In virtue of subsection 2.2, this equation, written in relative geometry, reads as

$$V_{n,\phi} = -\kappa_\phi \quad \text{ on } \ \Gamma_t, \tag{3.7}$$

that is the interface motion equation (P^0), whereas, in the Euclidian geometry, the same equation reads as

$$\frac{m(x)}{\sqrt{2a(x,n)}} \, V_n = - \operatorname{div} \Big[\frac{m(x)}{\sqrt{2a(x,n)}} \, a_p(x,n) \Big] \quad \text{ on } \ \Gamma_t. \tag{3.8}$$

Summarizing, under the assumption that the solution u^ε of Problem (P^ε) satisfies

$$u^\varepsilon \to \begin{cases} 1 & \text{in } Q_T^+ \\ 0 & \text{in } Q_T^- \end{cases} \qquad \text{as } \ \varepsilon \to 0,$$

we have formally proved that the boundary Γ_t between Ω_t^- and Ω_t^+ moves according to the law (3.7) or (3.8).

Remark 3.2. To conclude this section, note that (3.6) now yields $U_1 = 0$. In fact, the second term of the asymptotic expansion vanishes because the two stable zeros of the nonlinearity f have "balanced" stability, or more precisely because of the assumption $\int_0^1 f(u)du = 0$. If we perturb the nonlinearity by order ε, say $f(u) \longleftarrow f(u) - \varepsilon g(x,t,u)$, the equation in the free boundary problem contains an additional driving force term and U_1 no longer vanishes. More precisely, the equation will read as

$$V_{n,\phi} = -\kappa_\phi + c_0 \int_0^1 g(x,t,r)dr \quad \text{ on } \Gamma_t,$$

with c_0 a constant explicitly determined by the nonlinearity f. We refer to [1] for details. □

4 A comparison principle

This section is devoted to a comparison principle for weak solutions of Problem (P^ε). Such a result is rather standard (see [8]), but, since the problem is non-regular where $\nabla u = 0$, we prove it here for the self-containedness of the paper.

To begin with, we define a notion of sub- and super-solution of Problem (P^ε).

Definition 4.1. A function $u_\varepsilon^+ \in L^2(0,T;H^1(\Omega)) \cap L^\infty(Q_T)$ is a weak super-solution of Problem (P^ε), if

(i) $(u_\varepsilon^+)_t \in L^2(Q_T)$,

(ii) $\nabla_\phi u_\varepsilon^+(x,t) = a_p(x, \nabla u_\varepsilon^+(x,t)) \in L^\infty(0,T;L^2(\Omega))$,

(iii) u^ε satisfies the integral inequality

$$\int_0^t \int_\Omega \Big[(u_\varepsilon^+)_t\varphi + a_p(x,\nabla u_\varepsilon^+)\cdot\nabla\varphi - \frac{1}{\varepsilon^2}f(u_\varepsilon^+)\varphi\Big]m(x)dxdt \geq 0, \quad (4.1)$$

for all nonnegative function $\varphi \in L^2(0,T;H^1(\Omega)) \cap L^\infty(Q_T)$ and for all $t \in (0,T)$.

We define a weak sub-solution u_ε^- in a similar way, by changing $\geq$ in (4.1) by $\leq$.

The following remark will be useful when constructing smooth sub- and super-solutions in later sections.

Remark 4.2. If $u_\varepsilon^+ \in C^{2,1}(Q_T)$, it is not difficult to see that u_ε^+ is a super-solution in the sense defined above if and only if

(i) $a_p(x,\nabla u_\varepsilon^+)\cdot\nu \geq 0 \quad \text{on } \partial\Omega\times(0,T)$,

(ii) $\mathcal{L}_0 u_\varepsilon^+ \geq 0 \quad$ almost everywhere in Q_T,

where the operator $\mathcal{L}_0$ is defined by

$$\mathcal{L}_0 u := u_t - \frac{1}{m(x)}\operatorname{div}\Big[m(x)a_p(x,\nabla u)\Big] - \frac{1}{\varepsilon^2}f(u) = u_t - \Delta_\phi u - \frac{1}{\varepsilon^2}f(u).$$

In fact, if $u_\varepsilon^+ \in C^{2,1}(Q_T)$ then, by Lemma 2.2, the function $\mathcal{L}_0 u_\varepsilon^+$ is well-defined in Q_T. Also, using Lemma 2.2, we deduce that $\Delta_\phi u_\varepsilon^+ \in L^\infty(Q_T)$. The statement is then obtained by integrating (4.1) by parts. An analogous remark stands for a sub-solution $u_\varepsilon^- \in C^{2,1}(Q_T)$. □

We prove below an inequality which expresses the strong monotonicity of the function $T^0(x,p) = a_p(x,p)$.

Lemma 4.3. *There exists a constant $\beta > 0$ such that, for all $x \in \overline{\Omega}$, for all $p_1, p_2 \in \mathbb{R}^N$,*

$$(a_p(x,p_2) - a_p(x,p_1)) \cdot (p_2 - p_1) \geq \beta |p_2 - p_1|^2. \tag{4.2}$$

Proof. First we consider the case that $sp_1 + (1-s)p_2 \neq 0$ for all $s \in [0,1]$. Then, the function $s \mapsto a(x, sp_1 + (1-s)p_2)$ is of class C^2 on $[0,1]$ and there exist p_3, p_4 on the line segment $[p_1, p_2]$ such that

$$a(x,p_2) - a(x,p_1) = a_p(x,p_1) \cdot (p_2 - p_1) + \frac{1}{2}(p_2 - p_1) \cdot a_{pp}(x,p_3)(p_2 - p_1),$$

$$a(x,p_1) - a(x,p_2) = a_p(x,p_2) \cdot (p_1 - p_2) + \frac{1}{2}(p_1 - p_2) \cdot a_{pp}(x,p_4)(p_1 - p_2).$$

The strict convexity of $a(x,\cdot)$ implies that $a_{pp}(x,p)$ is a positively definite symmetric matrix, so that the function $(x,p,\bar{p}) \mapsto a_{pp}(x,p)\bar{p} \cdot \bar{p}$ is strictly positive and continuous on the compact set $\overline{\Omega} \times S^{N-1} \times S^{N-1}$. Hence there exist constants $0 < \lambda_2 \leq \Lambda_2$ such that, for all $x \in \overline{\Omega}$, all $p \in \mathbb{R}^N \setminus \{0\}$, all $\bar{p} \in \mathbb{R}^N$,

$$\lambda_2 |\bar{p}|^2 \leq a_{pp}(x,p)\bar{p} \cdot \bar{p} \leq \Lambda_2 |\bar{p}|^2. \tag{4.3}$$

It then follows that

$$a(x,p_2) - a(x,p_1) \geq a_p(x,p_1) \cdot (p_2 - p_1) + \frac{\lambda_2}{2}|p_2 - p_1|^2, \tag{4.4}$$

$$a(x,p_1) - a(x,p_2) \geq a_p(x,p_2) \cdot (p_1 - p_2) + \frac{\lambda_2}{2}|p_2 - p_1|^2. \tag{4.5}$$

Adding up inequalities (4.4) and (4.5) yields the desired inequality, with the constant $\beta = \lambda_2$.

In the case that $sp_1 + (1-s)p_2 = 0$ for some $s \in [0,1]$, p_1 and p_2 are colinear and we may suppose that there exists $l \in \mathbb{R}$ such that $p_2 = lp_1$. We can assume $l \neq 0$, $l \neq 1$ and $p_1 \neq 0$. By using the properties stated in Remark 1.2, we obtain that

$$\begin{aligned}
(a_p(x,p_2) - a_p(x,p_1)) \cdot (p_2 - p_1) &= (l-1)^2 a_p(x,p_1) \cdot p_1 \\
&= 2(l-1)^2 a(x,p_1) \\
&= 2a(x,(l-1)p_1) \\
&\geq {\lambda_0}^2 |(l-1)p_1|^2 = {\lambda_0}^2 |p_2 - p_1|^2,
\end{aligned}$$

where λ_0 has been defined in (2.16). The proof is now completed. $\square$

We are now ready to prove the following comparison principle.

Proposition 4.4 (Comparison principle)**.** *Let u_ε^+, respectively u_ε^-, be a super-solution, respectively a sub-solution, of Problem (P^ε). Assume that*

$$u_\varepsilon^-(\cdot,0) \leq u_\varepsilon^+(\cdot,0) \quad \textit{almost everywhere in } \Omega.$$

Then we have that

$$u_\varepsilon^- \le u^\varepsilon \le u_\varepsilon^+ \quad \textit{almost everywhere in } Q_T.$$

Proof. By subtracting inequality (4.1) for the super-solution u_ε^+ from inequality for the sub-solution u_ε^-, we obtain that, for all $\varphi \in L^2(0,T;H^1(\Omega)) \cap L^\infty(Q_T)$ such that $\varphi \ge 0$, and for all $t \in (0,T)$,

$$\int_0^t \int_\Omega \Big[(u_\varepsilon^- - u_\varepsilon^+)_t \varphi + (a_p(x,\nabla u_\varepsilon^-) - a_p(x,\nabla u_\varepsilon^+)) \cdot \nabla\varphi\Big] m(x)$$
$$\le C \int_0^t \int_\Omega |u_\varepsilon^- - u_\varepsilon^+|\varphi\,, \tag{4.6}$$

where C is a constant depending on ε and the L^∞ norms of f' and m. Next we set $\varphi = (u_\varepsilon^- - u_\varepsilon^+)^+$, which belongs to $L^2(0,T;H^1(\Omega)) \cap L^\infty(Q_T)$; it follows from (4.2) that

$$\begin{aligned}
&\int_0^t \int_\Omega (a_p(x,\nabla u_\varepsilon^-) - a_p(x,\nabla u_\varepsilon^+)) \cdot \nabla\varphi\, m(x) \\
&\qquad = \int_0^t \int_{\{u_\varepsilon^- - u_\varepsilon^+ \ge 0\}} (a_p(x,\nabla u_\varepsilon^-) - a_p(x,\nabla u_\varepsilon^+)) \cdot (\nabla u_\varepsilon^- - \nabla u_\varepsilon^+) m(x) \\
&\qquad \ge m_1 \beta \int_0^t \int_{\{u_\varepsilon^- - u_\varepsilon^+ \ge 0\}} |\nabla u_\varepsilon^- - \nabla u_\varepsilon^+|^2 \ge 0.
\end{aligned}$$

In view of (4.6), we now have that

$$\frac{m_1}{2} \int_0^t \frac{d}{dt} \int_\Omega \Big((u_\varepsilon^- - u_\varepsilon^+)^+\Big)^2 \le C \int_0^t \int_{\{u_\varepsilon^- - u_\varepsilon^+ \ge 0\}} (u_\varepsilon^- - u_\varepsilon^+)^2\,,$$

and therefore

$$\int_\Omega \Big((u_\varepsilon^- - u_\varepsilon^+)^+\Big)^2(t) \le \frac{2C}{m_1} \int_0^t \int_\Omega \Big((u_\varepsilon^- - u_\varepsilon^+)^+\Big)^2 + \int_\Omega \Big((u_\varepsilon^- - u_\varepsilon^+)^+\Big)^2(0).$$

Gronwall's lemma yields

$$\int_\Omega \Big((u_\varepsilon^- - u_\varepsilon^+)^+\Big)^2(t) \le e^{2Ct/m_1} \int_\Omega \Big((u_\varepsilon^- - u_\varepsilon^+)^+\Big)^2(0).$$

Since $u_\varepsilon^-(x,0) \le u_\varepsilon^+(x,0)$ for almost all $x \in \Omega$, it follows that

$$u_\varepsilon^- \le u_\varepsilon^+ \quad \text{a.e. in } Q_T.$$

□

Lemma 4.5. *Let u^ε be the solution of Problem (P^ε) (with initial data u_0). Then*

$$-\|u_0\|_{L^\infty(\Omega)} \le u^\varepsilon \le \max(1, \|u_0\|_{L^\infty(\Omega)}) \quad \textit{a.e. in } Q_T.$$

Proof. By the bistable profile of f, we remark that $-\|u_0\|_{L^\infty(\Omega)}$, respectively $\max(1, \|u_0\|_{L^\infty(\Omega)})$, is a sub-solution, respectively a super-solution, of Problem (P^ε). □

5 Generation of the interface

This section deals with the generation of the interface, namely the rapid formation of internal layers that takes place in a neighborhood of $\Gamma_0 = \{x \in \Omega,\, u_0(x) = a\}$ within the time span of order $\varepsilon^2|\ln\varepsilon|$. In the sequel, η_0 will stand for the quantity

$$\eta_0 := \min(a, 1-a).$$

Our main result in this section is the following.

Theorem 5.1. *Let $\eta \in (0, \eta_0)$ be arbitrary and define μ as the derivative of $f(u)$ at the unstable equilibrium $u = a$, that is*

$$\mu = f'(a). \tag{5.1}$$

Then there exist positive constants ε_0 and M_0 such that, for all $\varepsilon \in (0, \varepsilon_0)$,

(i) *for almost all $x \in \Omega$,*

$$-\eta \le u^\varepsilon(x, \mu^{-1}\varepsilon^2|\ln\varepsilon|) \le 1 + \eta\,, \tag{5.2}$$

(ii) *for almost all $x \in \Omega$ such that $|u_0(x) - a| \ge M_0\varepsilon$, we have that*

$$\text{if} \quad u_0(x) \ge a + M_0\varepsilon \quad \text{then} \quad u^\varepsilon(x, \mu^{-1}\varepsilon^2|\ln\varepsilon|) \ge 1 - \eta, \tag{5.3}$$

$$\text{if} \quad u_0(x) \le a - M_0\varepsilon \quad \text{then} \quad u^\varepsilon(x, \mu^{-1}\varepsilon^2|\ln\varepsilon|) \le \eta. \tag{5.4}$$

We will prove this result by constructing a suitable pair of sub and super-solutions.

5.1 The bistable ordinary differential equation

The sub- and super-solutions mentioned above will be constructed by modifying the solution of the problem without diffusion:

$$\bar{u}_t = \frac{1}{\varepsilon^2} f(\bar{u}), \qquad \bar{u}(x, 0) = u_0(x).$$

This solution is written in the form

$$\bar{u}(x,t) = Y\Big(\frac{t}{\varepsilon^2},\, u_0(x)\Big),$$

where $Y(\tau, \xi)$ denotes the solution of the ordinary differential equation

$$\begin{cases} Y_\tau(\tau,\xi) &= f(Y(\tau,\xi)) \quad \text{for } \tau > 0 \\ Y(0,\xi) &= \xi. \end{cases} \tag{5.5}$$

Here ξ ranges over the interval $(-2C_0, 2C_0)$, with C_0 being the constant defined in (1.7). We first collect basic properties of Y.

Lemma 5.2. *We have $Y_\xi > 0$, for all $\xi \in (-2C_0, 2C_0)$ and all $\tau > 0$. Furthermore,*

$$Y_\xi(\tau,\xi) = \frac{f(Y(\tau,\xi))}{f(\xi)}.$$

Proof. First, differentiating equation (5.5) with respect to ξ, we obtain

$$\begin{cases} Y_{\xi\tau} = Y_\xi f'(Y), \\ Y_\xi(0,\xi) = 1, \end{cases} \tag{5.6}$$

which can be integrated as follows:

$$Y_\xi(\tau,\xi) = \exp\Big[\int_0^\tau f'(Y(s,\xi))ds\Big] > 0. \tag{5.7}$$

We then differentiate equation (5.5) with respect to τ and obtain

$$\begin{cases} Y_{\tau\tau} = Y_\tau f'(Y), \\ Y_\tau(0,\xi) = f(\xi), \end{cases} \tag{5.8}$$

which in turn implies

$$Y_\tau(\tau,\xi) = f(\xi)\exp\Big[\int_0^\tau f'(Y(s,\xi))ds\Big] = f(\xi)Y_\xi(\tau,\xi). \tag{5.9}$$

This last equality, in view of (5.5), completes the proof of Lemma 5.2. $\square$

We define a function $A(\tau,\xi)$ by

$$A(\tau,\xi) = \frac{f'(Y(\tau,\xi)) - f'(\xi)}{f(\xi)}. \tag{5.10}$$

Lemma 5.3. *We have, for all $\xi \in (-2C_0, 2C_0)$ and all $\tau > 0$,*

$$A(\tau,\xi) = \int_0^\tau f''(Y(s,\xi))Y_\xi(s,\xi)ds.$$

Proof. Differentiating the equality of Lemma 5.2 with respect to ξ leads to

$$Y_{\xi\xi} = A(\tau,\xi)Y_\xi, \tag{5.11}$$

whereas differentiating (5.7) with respect to ξ yields

$$Y_{\xi\xi} = Y_\xi \int_0^\tau f''(Y(s,\xi))Y_\xi(s,\xi)ds.$$

These two last results complete the proof of Lemma 5.3. $\square$

Next we need some estimates on Y and its derivatives. First, we perform some estimates when the initial value ξ lies between η and $1-\eta$.

Lemma 5.4. *Let $\eta \in (0, \eta_0)$ be arbitrary. Then there exist positive constants $\tilde{C}_1 = \tilde{C}_1(\eta)$, $\tilde{C}_2 = \tilde{C}_2(\eta)$ and $C_3 = C_3(\eta)$ such that, for all $\tau > 0$,*

(i) *if $\xi \in (a, 1-\eta)$ then, for every $\tau > 0$ such that $Y(\tau, \xi)$ remains in the interval $(a, 1-\eta)$, we have*

$$\tilde{C}_1 e^{\mu\tau} \leq Y_\xi(\tau, \xi) \leq \tilde{C}_2 e^{\mu\tau}\,, \tag{5.12}$$

and

$$|A(\tau, \xi)| \leq C_3(e^{\mu\tau} - 1); \tag{5.13}$$

(ii) *if $\xi \in (\eta, a)$ then, for every $\tau > 0$ such that $Y(\tau, \xi)$ remains in the interval (η, a),* (5.12) *and* (5.13) *hold as well,*

where μ is the constant defined in (5.1).

Proof. We take $\xi \in (a, 1-\eta)$ and suppose that for $s \in (0, \tau)$, $Y(s, \xi)$ remains in the interval $(a, 1-\eta)$. Integrating the equality $Y_\tau / f(Y) = 1$ from 0 to τ yields

$$\tau = \int_0^\tau \frac{Y_\tau(s, \xi)}{f(Y(s, \xi))} ds = \int_\xi^{Y(\tau,\xi)} \frac{dq}{f(q)}. \tag{5.14}$$

Moreover, the equality of Lemma 5.2 leads to

$$\begin{aligned} \ln Y_\xi(\tau, \xi) &= \int_\xi^{Y(\tau,\xi)} \frac{f'(q)}{f(q)} dq \\ &= \int_\xi^{Y(\tau,\xi)} [\frac{f'(a)}{f(q)} + \frac{f'(q) - f'(a)}{f(q)}] dq \\ &= \mu\tau + \int_\xi^{Y(\tau,\xi)} h(q) dq, \end{aligned} \tag{5.15}$$

where

$$h(q) = (f'(q) - \mu)/f(q).$$

Since

$$h(q) \to \frac{f''(a)}{f'(a)} \quad \text{as} \quad q \to a,$$

the function h is continuous on $[a, 1-\eta]$. Hence we can define

$$H = H(\eta) := \|h\|_{L^\infty(a, 1-\eta)}.$$

Since $|Y(\tau, \xi) - \xi|$ takes its values in the interval $[0, 1-a-\eta] \subset [0, 1-a]$, it follows from (5.15) that

$$\mu\tau - H(1-a) \leq \ln Y_\xi(\tau, \xi) \leq \mu\tau + H(1-a),$$

which, in turn, proves (5.12). Lemma 5.3 and (5.12) yield

$$|A(\tau,\xi)| \leq \sup_{z\in[0,1]} |f''(z)| \int_0^\tau \tilde{C}_2 e^{\mu s} ds \leq C_3(e^{\mu\tau} - 1),$$

which completes the proof of (5.13). The case where ξ and $Y(\tau,\xi)$ are in (η, a) is similar and omitted. □

Corollary 5.5. *Let $\eta \in (0,\eta_0)$ be arbitrary. Then there exist positive constants $C_1 = C_1(\eta)$ and $C_2 = C_2(\eta)$ such that, for all $\tau > 0$,*

(i) *if $\xi \in (a, 1-\eta)$ then, for every $\tau > 0$ such that $Y(\tau,\xi)$ remains in the interval $(a, 1-\eta)$, we have*

$$C_1 e^{\mu\tau}(\xi - a) \leq Y(\tau,\xi) - a \leq C_2 e^{\mu\tau}(\xi - a); \tag{5.16}$$

(ii) *if $\xi \in (\eta, a)$ then, for every $\tau > 0$ such that $Y(\tau,\xi)$ remains in the interval (η, a), we have*

$$C_2 e^{\mu\tau}(\xi - a) \leq Y(\tau,\xi) - a \leq C_1 e^{\mu\tau}(\xi - a). \tag{5.17}$$

Proof. Since

$$f(q)/(q-a) \to f'(a) \quad \text{ as } \; q \to a,$$

it is possible to find $B_1 = B_1(\eta) > 0$ and $B_2 = B_2(\eta) > 0$ such that, for all $q \in (a, 1-\eta)$,

$$B_1(q-a) \leq f(q) \leq B_2(q-a). \tag{5.18}$$

We write this inequality for $a < Y(\tau,\xi) < 1-\eta$ to obtain

$$B_1(Y(\tau,\xi) - a) \leq f(Y(\tau,\xi)) \leq B_2(Y(\tau,\xi) - a).$$

We also write this inequality for $a < \xi < 1-\eta$ to obtain

$$B_1(\xi - a) \leq f(\xi) \leq B_2(\xi - a).$$

Next we use the equality $Y_\xi = f(Y)/f(\xi)$ of Lemma 5.2 to deduce that

$$\frac{B_1}{B_2}(Y(\tau,\xi) - a) \leq (\xi - a) Y_\xi(\tau,\xi) \leq \frac{B_2}{B_1}(Y(\tau,\xi) - a),$$

which, in view of (5.12), implies that

$$\frac{B_1}{B_2}\tilde{C}_1 e^{\mu\tau}(\xi - a) \leq Y(\tau,\xi) - a \leq \frac{B_2}{B_1}\tilde{C}_2 e^{\mu\tau}(\xi - a).$$

This proves (5.16). The proof of (5.17) is similar and omitted. □

Next we present estimates in the case where the initial value ξ is smaller than η or larger than $1-\eta$.

Lemma 5.6. *Let $\eta \in (0, \eta_0)$ and $M > 0$ be arbitrary. Then there exists a positive constant $C_4 = C_4(\eta, M)$ such that*

(i) *if $\xi \in [1-\eta, 1+M]$, then, for all $\tau > 0$, $Y(\tau, \xi)$ remains in the interval $[1-\eta, 1+M]$ and*

$$|A(\tau, \xi)| \le C_4 \tau \quad \textit{for} \quad \tau > 0\,; \tag{5.19}$$

(ii) *if $\xi \in [-M, \eta]$, then, for all $\tau > 0$, $Y(\tau, \xi)$ remains in the interval $[-M, \eta]$ and* (5.19) *holds as well.*

Proof. Since the two statements can be treated in the same way, we will only prove the former. The fact that $Y(\tau, \xi)$, the solution of the ordinary differential equation (5.5), remains in the interval $[1-\eta, 1+M]$ directly follows from the bistable properties of f, or, more precisely, from the sign conditions $f(1-\eta) > 0$, $f(1+M) < 0$.

To prove (5.19), suppose first that $\xi \in [1, 1+M]$. In view of (1.3), f' is strictly negative in an interval of the form $[1, 1+c]$ and f is negative in $[1, \infty)$. We denote by $-m < 0$ the maximum of f on $[1+c, 1+M]$. Then, as long as $Y(\tau, \xi)$ remains in the interval $[1+c, 1+M]$, the ordinary differential equation (5.5) implies

$$Y_\tau \le -m.$$

By integration, this means that, for any $\xi \in [1, 1+M]$, we have

$$Y(\tau, \xi) \in [1, 1+c] \qquad \text{for} \quad \tau \ge \overline{\tau} := \frac{M-c}{m}.$$

In view of this, and considering that $f'(Y) < 0$ for $Y \in [1, 1+c]$, we see from the expression (5.7) that

$$\begin{aligned} Y_\xi(\tau, \xi) &= \exp\Big[\int_0^{\overline{\tau}} f'(Y(s,\xi))ds\Big] \exp\Big[\int_{\overline{\tau}}^{\tau} f'(Y(s,\xi))ds\Big] \\ &\le \exp\Big[\int_0^{\overline{\tau}} f'(Y(s,\xi))ds\Big] \\ &\le \exp\Big[\int_0^{\overline{\tau}} \sup_{z \in [-M, 1+M]} |f'(z)| ds\Big] =: \tilde{C}_4 = \tilde{C}_4(M), \end{aligned}$$

for all $\tau \ge \overline{\tau}$. It is clear from the same expression (5.7) that $Y_\xi \le \tilde{C}_4$ holds also for $0 \le \tau \le \overline{\tau}$. We can then use Lemma 5.3 to deduce that

$$\begin{aligned} |A(\tau, \xi)| &\le \tilde{C}_4 \int_0^\tau |f''(Y(s,\xi))| ds \\ &\le \tilde{C}_4 \Big(\sup_{z \in [-M, 1+M]} |f''(z)| \Big) \tau =: C_4 \tau. \end{aligned}$$

The case $\xi \in [1-\eta, 1]$ can be treated in the same way. This completes the proof of the lemma. $\square$

Now we choose the constant M in the above lemma sufficiently large so that $[-2C_0, 2C_0] \subset [-M, 1+M]$, and fix M hereafter. Then C_4 only depends on η. Using the fact that $\tau = O(e^{\mu\tau} - 1)$ for $\tau > 0$, one can easily deduce from (5.13) and (5.19) the following general estimate.

Lemma 5.7. *Let $\eta \in (0, \eta_0)$ be arbitrary and let C_0 be the constant defined in (1.7). Then there exists a positive constant $C_5 = C_5(\eta)$ such that, for all $\xi \in (-2C_0, 2C_0)$ and all $\tau > 0$,*

$$|A(\tau, \xi)| \le C_5(e^{\mu\tau} - 1).$$

5.2 Construction of sub- and super-solutions

We are now ready to construct sub- and super-solutions in order to study the generation of the interface. By using some cut-off initial data, see subsection 3.2 in [1], we can modify slightly u_0 near the boundary $\partial\Omega$ and make, without loss of generality, the additional assumption

$$a_p(x, \nabla u_0(x)) \cdot \nu = 0 \quad \text{ on } \partial\Omega. \tag{5.20}$$

Our sub- and super-solutions are defined by

$$w_\varepsilon^\pm(x,t) = Y\Big(\frac{t}{\varepsilon^2},\, u_0(x) \pm \varepsilon^2 C_6(e^{\mu t/\varepsilon^2} - 1)\Big). \tag{5.21}$$

Lemma 5.8. *There exist positive constants ε_0 and C_6 such that, for all $\varepsilon \in (0, \varepsilon_0)$, $(w_\varepsilon^-, w_\varepsilon^+)$ is a pair of sub- and super-solutions of Problem (P^ε), in the domain $\Omega \times (0, \mu^{-1}\varepsilon^2 |\ln \varepsilon|)$.*

Proof. Following Remark 4.2 we define the operator $\mathcal{L}_0$ by

$$\mathcal{L}_0 u := u_t - \frac{1}{m(x)} \operatorname{div}\Big[m(x) a_p(x, \nabla u)\Big] - \frac{1}{\varepsilon^2} f(u)\,, \tag{5.22}$$

and prove that $\mathcal{L}_0 w_\varepsilon^+ \ge 0$. We compute

$$(w_\varepsilon{}^+)_t = \frac{1}{\varepsilon^2} Y_\tau + \mu C_6 e^{\mu t/\varepsilon^2} Y_\xi,$$
$$\nabla w_\varepsilon^+ = \nabla u_0(x) Y_\xi.$$

Using (5.20) and the fact that $a_p(x, \cdot)$ is homogeneous of degree one, we see that $w_\varepsilon^\pm$ satisfy the anisotropic Neumann boundary condition $a_p(x, \nabla w_\varepsilon^\pm) \cdot \nu = 0$ on $\partial\Omega \times (0, +\infty)$. In view of the ordinary differential equation (5.5), we obtain

$$\mathcal{L}_0 w_\varepsilon^+ = \mu C_6 e^{\mu t/\varepsilon^2} Y_\xi - \frac{1}{m(x)} \operatorname{div}\Big[m(x) a_p(x, \nabla w_\varepsilon^+)\Big].$$

By the estimate of the anisotropic Laplacian (2.26), it follows that

$$\mathcal{L}_0 w_\varepsilon^+ \geq \mu C_6 e^{\mu t/\varepsilon^2} Y_\xi - C_L(|\nabla w_\varepsilon^+(x,t)| + |D^2 w_\varepsilon^+(x,t)|)\,, \tag{5.23}$$

where we recall that $|D^2 w_\varepsilon^+(x,t)| = \max_{i,j} |\partial_i \partial_j w_\varepsilon^+(x,t)|$. A straightforward calculation yields

$$\partial_i \partial_j w_\varepsilon^+(x,t) = (\partial_i \partial_j u_0) Y_\xi + (\partial_i u_0 \partial_j u_0) Y_{\xi\xi}.$$

Recalling that $Y_\xi > 0$, we now combine the expression of ∇w_ε^+, the above expression and inequality (5.23) to obtain

$$\mathcal{L}_0 w_\varepsilon^+ / Y_\xi \geq \mu C_6 e^{\mu t/\varepsilon^2} - C_L C_0 - C_0 - {C_0}^2 \frac{|Y_{\xi\xi}|}{Y_\xi}\,, \tag{5.24}$$

where C_0 is the constant defined in (1.7). We note that, in the range $(0, \mu^{-1}\varepsilon^2 |\ln \varepsilon|)$, we have

$$0 \leq \varepsilon^2 C_6 (e^{\mu t/\varepsilon^2} - 1) \leq \varepsilon^2 C_6 (\varepsilon^{-1} - 1) \leq C_0\,,$$

if ε_0 is small enough. Hence

$$\xi := u_0(x) \pm \varepsilon^2 \varepsilon C_6 (e^{\mu t/\varepsilon^2} - 1) \in (-2C_0, 2C_0),$$

so that, by the results of the previous subsection, Y remains in $(-2C_0, 2C_0)$. In view of (5.11), $Y_{\xi\xi}/Y_\xi$ is equal to A so that, combining the estimate of A in Lemma 5.7 and (5.24), we obtain

$$\mathcal{L}_0 w_\varepsilon^+ / Y_\xi \geq (\mu C_6 - {C_0}^2 C_5) e^{\mu t/\varepsilon^2} - C_L C_0 - C_0.$$

Now, choosing

$$C_6 \geq \frac{2}{\mu} \max\left({C_0}^2 C_5, C_0(C_L + 1)\right)$$

proves $\mathcal{L}_0 w_\varepsilon^+ / Y_\xi \geq 0$. Since $Y_\xi > 0$, it follows that $\mathcal{L}_0 w_\varepsilon^+ \geq 0$. Hence, by Remark 4.2, w_ε^+ is a super-solution of Problem (P^ε). Similarly w_ε^- is a sub-solution. Lemma 5.8 is proved. $\square$

To conclude this subsection, we remark that $w^\pm(x,0) = Y\Big(\frac{t}{\varepsilon^2}, u_0(x)\Big) = u_0(x)$. Consequently, by the comparison principle,

$$w_\varepsilon^-(x,t) \leq u^\varepsilon(x,t) \leq w_\varepsilon^+(x,t)\,, \tag{5.25}$$

for almost all $(x,t) \in \Omega \times (0, \mu^{-1}\varepsilon^2 |\ln \varepsilon|)$.

5.3 Proof of Theorem 5.1

In order to prove Theorem 5.1 we first present a key estimate on the function Y after a time interval of order $\tau \sim |\ln \varepsilon|$.

Lemma 5.9. *Let $\eta \in (0, \eta_0)$ be arbitrary; there exist positive constants ε_0 and C_7 such that, for all $\varepsilon \in (0, \varepsilon_0)$,*

(i) *for all $\xi \in (-2C_0, 2C_0)$, for all $\tau \geq \mu^{-1}|\ln \varepsilon|$,*

$$-\eta \leq Y(\tau, \xi) \leq 1 + \eta\,, \tag{5.26}$$

(ii) *for all $\xi \in (-2C_0, 2C_0)$ such that $|\xi - a| \geq C_7\varepsilon$, for all $\tau \geq \mu^{-1}|\ln \varepsilon|$,*

$$\textit{if} \quad \xi \geq a + C_7\varepsilon \quad \textit{then} \quad Y(\tau, \xi) \geq 1 - \eta, \tag{5.27}$$

$$\textit{if} \quad \xi \leq a - C_7\varepsilon \quad \textit{then} \quad Y(\tau, \xi) \leq \eta. \tag{5.28}$$

Proof. We first prove (5.27). For $\xi \geq a + C_7\varepsilon$, as long as $Y(\tau, \xi)$ has not reached $1 - \eta$, we can use (5.16) to deduce that

$$Y(\tau, \xi) \geq a + C_1 e^{\mu\tau}(\xi - a) \geq a + C_1 C_7 e^{\mu\tau}\varepsilon \geq 1 - \eta\,,$$

provided that τ satisfies $\tau \geq \mu^{-1} \ln \frac{1-a-\eta}{C_1 C_7 \varepsilon}$. Choosing

$$C_7 = \frac{\max(a, 1-a) - \eta}{C_1}$$

completes the proof of (5.27). Using (5.17), one easily proves (5.28).

Next we prove (5.26). First, by the bistable assumptions on f, if we leave from a $\xi \in [-\eta, 1 + \eta]$ then $Y(\tau, \xi)$ will remain in $[-\eta, 1 + \eta]$. Now suppose that $1 + \eta \leq \xi \leq 2C_0$. We check below that $Y(\mu^{-1}|\ln \varepsilon|, \xi) \leq 1 + \eta$. First, in view of (1.3), we can find $p > 0$ such that

$$\begin{array}{llll} \text{if} & 1 \leq u \leq 2C_0 & \text{then} & f(u) \leq p(1 - u), \\ \text{if} & -2C_0 \leq u \leq 0 & \text{then} & f(u) \geq -pu. \end{array} \tag{5.29}$$

We then use the ordinary differential equation (5.5) to obtain, as long as $1 + \eta \leq Y \leq 2C_0$, the inequality $Y_\tau \leq p(1 - Y)$. It follows that

$$\frac{Y_\tau}{Y - 1} \leq -p.$$

Integrating this inequality from 0 to τ leads to

$$Y(\tau, \xi) \leq 1 + (\xi - 1)e^{-p\tau} \leq 1 + (2C_0 - 1)e^{-p\tau}.$$

One easily checks that, for $\varepsilon \in (0, \varepsilon_0)$, with $\varepsilon_0 = \varepsilon_0(\eta)$ small enough, we have $Y(\tau, \xi) \leq 1 + \eta$, for all $\tau \geq \mu^{-1}|\ln \varepsilon|$, which completes the proof of (5.26). $\square$

We are now ready to prove Theorem 5.1. By setting $t = \mu^{-1}\varepsilon^2|\ln\varepsilon|$ in (5.25), we obtain, for almost all $x \in \Omega$,

$$\begin{aligned} &Y(\mu^{-1}|\ln\varepsilon|, u_0(x) - (C_6\varepsilon - C_6\varepsilon^2)) \\ &\qquad \leq u^\varepsilon(x, \mu^{-1}\varepsilon^2|\ln\varepsilon|) \leq Y(\mu^{-1}|\ln\varepsilon|, u_0(x) + C_6\varepsilon - C_6\varepsilon^2). \end{aligned} \tag{5.30}$$

Furthermore, by the definition of C_0 in (1.7), we have, for ε_0 small enough,

$$-2C_0 \leq u_0(x) \pm (C_6\varepsilon - C_6\varepsilon^2) \leq 2C_0 \qquad \text{for } x \in \Omega.$$

Thus the assertion (5.2) of Theorem 5.1 is a direct consequence of (5.26) and (5.30).

Next we prove (5.3). We choose M_0 large enough so that $M_0\varepsilon - C_6\varepsilon + C_6\varepsilon^2 \geq C_7\varepsilon$. Then, for any $x \in \Omega$ such that $u_0(x) \geq a + M_0\varepsilon$, we have

$$u_0(x) - (C_6\varepsilon - C_6\varepsilon^2) \geq a + M_0\varepsilon - C_6\varepsilon + C_6\varepsilon^2 \geq a + C_7\varepsilon.$$

Combining this, (5.30) and (5.27), we see that

$$u^\varepsilon(x, \mu^{-1}\varepsilon^2|\ln\varepsilon|) \geq 1 - \eta\,,$$

for almost all $x \in \Omega$ that satisfies $u_0(x) \geq a + M_0\varepsilon$. This proves (5.3). The inequality (5.4) can be shown the same way. This completes the proof of Theorem 5.1. $\square$

6 Motion of the interface

We have seen in Section 5 that, after a very short time, the solution u^ε develops a clear transition layer. In the present section, we show that it persists and that its law of motion is well approximated by the interface equation (P^0).

More precisely, take the first term of the formal asymptotic expansion (3.2) as a formal expansion of the solution:

$$u^\varepsilon(x,t) \;\approx\; \tilde{u}^\varepsilon(x,t) := U_0\left(\frac{\widetilde{d}_\phi(x,t)}{\varepsilon}\right). \tag{6.1}$$

The right-hand side of (6.1) is a function having a well-developed transition layer, and its interface lies exactly on Γ_t. We show that this function is a very good approximation of the solution; therefore the following holds:

> *If u^ε becomes rather close to $\tilde{u}^\varepsilon$ at some time moment, then it stays close to $\tilde{u}^\varepsilon$ for the rest of time.*

To that purpose, we will construct a pair of sub- and super-solutions u_ε^- and u_ε^+ of Problem (P^ε) by slightly modifying $\tilde{u}^\varepsilon$. It then follows that, if the solution u^ε satisfies

$$u_\varepsilon^-(x,t_0) \leq u^\varepsilon(x,t_0) \leq u_\varepsilon^+(x,t_0)\,,$$

for some $t_0 \geq 0$ and for almost all $x \in \Omega$, then

$$u_\varepsilon^-(x,t) \leq u^\varepsilon(x,t) \leq u_\varepsilon^+(x,t)\,,$$

for almost $(x,t) \in Q_T$ that satisfies $t_0 \leq t \leq T$. As a result, since both $u_\varepsilon^+, u_\varepsilon^-$ stay close to $\tilde{u}^\varepsilon$, the solution u^ε also stays close to $\tilde{u}^\varepsilon$ for $t_0 \leq t \leq T$.

6.1 Construction of sub and super-solutions

To begin with we present a mathematical tool which is essential for the construction of sub and super-solutions.

A modified anisotropic signed distance function. Rather than working with the anisotropic signed distance function $\widetilde{d}_\phi$, defined in (2.19), we define a "cut-off anisotropic signed distance function" d_ϕ as follows. Choose $d_0 > 0$ small enough so that $\widetilde{d}_\phi(\cdot,\cdot)$ is smooth in the tubular neighborhood of Γ

$$\{(x,t) \in Q_T,\ |\widetilde{d}_\phi(x,t)| < 3d_0\},$$

and that

$$\mathrm{dist}_\phi(\Gamma_t, \partial\Omega) > 3d_0 \qquad \text{for all } t \in (0,T). \tag{6.2}$$

Next let $\zeta(s)$ be a smooth increasing function on $\mathbb{R}$ such that

$$\zeta(s) = \begin{cases} s & \text{if } \ |s| \leq d_0 \\ -2d_0 & \text{if } \ s \leq -2d_0 \\ 2d_0 & \text{if } \ s \geq 2d_0. \end{cases}$$

We define the cut-off anisotropic signed distance function d_ϕ by

$$d_\phi(x,t) = \zeta\big(\widetilde{d}_\phi(x,t)\big). \tag{6.3}$$

Note that, in view of (2.20),

$$2a(x, \nabla d_\phi(x,t)) = 1 \qquad \text{in a neighborhood of } \Gamma_t, \tag{6.4}$$

more precisely in the region $\{(x,t) \in Q_T, |d_\phi(x,t)| < d_0\}$. Moreover, in view of (6.2), we have

$$2a(x, \nabla d_\phi(x,t)) = 0 \qquad \text{far away from } \Gamma_t, \tag{6.5}$$

more precisely in the region $\{(x,t) \in Q_T, |d_\phi(x,t)| \geq 2d_0\}$. Furthermore, since the moving interface Γ satisfies Problem (P^0), an alternative equation for Γ is given by

$$(d_\phi)_t = \frac{1}{m(x)} \operatorname{div} \big[m(x) a_p(x, \nabla d_\phi)\big] \quad \text{ on } \Gamma_t. \tag{6.6}$$

Construction. We look for a pair of sub- and super-solutions $u_\varepsilon^\pm$ for (P^ε) of the form

$$u_\varepsilon^\pm(x,t) = U_0\Big(\frac{d_\phi(x,t) \pm \varepsilon p(t)}{\varepsilon}\Big) \pm q(t), \tag{6.7}$$

where U_0 is the solution of (3.4), and where

$$p(t) = -e^{-\beta t/\varepsilon^2} + e^{Lt} + K,$$
$$q(t) = \sigma(\beta e^{-\beta t/\varepsilon^2} + \varepsilon^2 L e^{Lt}).$$

Note that $q = \sigma\varepsilon^2 p_t$. It is clear from the definition of $u_\varepsilon^\pm$ that

$$\lim_{\varepsilon \to 0} u_\varepsilon^\pm(x,t) = \begin{cases} 1 & \text{for all } (x,t) \in Q_T^+ \\ 0 & \text{for all } (x,t) \in Q_T^-. \end{cases} \tag{6.8}$$

The main result of this section is the following.

Lemma 6.1. *There exist positive constants β, σ with the following properties. For any $K > 1$, we can find positive constants ε_0 and L such that, for any $\varepsilon \in (0, \varepsilon_0)$, the functions u_ε^- and u_ε^+ satisfy the anisotropic Neumann boundary condition and*

$$\mathcal{L}_0 u_\varepsilon^- \leq 0 \leq \mathcal{L}_0 u_\varepsilon^+,$$

in the range $\Omega \times (0,T)$, where the operator $\mathcal{L}_0$ has been defined in (5.22).

6.2 Proof of Lemma 6.1

We show below that

$$\mathcal{L}_0 u_\varepsilon^+ := (u_\varepsilon^+)_t - \frac{1}{m(x)} \operatorname{div} \big[m(x) a_p(x, \nabla u_\varepsilon^+)\big] - \frac{1}{\varepsilon^2} f(u_\varepsilon^+) \geq 0,$$

the proof of inequality $\mathcal{L}_0 u_\varepsilon^- \leq 0$ follows by similar arguments.

6.2.1 Computation of $\mathcal{L}_0 u_\varepsilon^+$

In the sequel, the function U_0 and its derivatives are taken at the point $(d_\phi(x,t)+\varepsilon p(t))/\varepsilon$. Straightforward computations yield

$$\begin{aligned}
(u_\varepsilon^+)_t &= (\frac{1}{\varepsilon}(d_\phi)_t + p_t){U_0}' + q_t,\\
\nabla u_\varepsilon^+ &= \frac{1}{\varepsilon}{U_0}'\nabla d_\phi,\\
\operatorname{div} a_p(x,\nabla u_\varepsilon^+) &= \frac{1}{\varepsilon^2}{U_0}''\nabla d_\phi \cdot a_p(x,\nabla d_\phi) + \frac{1}{\varepsilon}{U_0}' \operatorname{div} a_p(x,\nabla d_\phi)\\
&= \frac{1}{\varepsilon^2}{U_0}'' 2a(x,\nabla d_\phi) + \frac{1}{\varepsilon}{U_0}' \operatorname{div} a_p(x,\nabla d_\phi),
\end{aligned}$$

where we have used properties stated in Remark 1.2. Note that, d_ϕ being constant in a neighborhood of $\partial\Omega$, we have that $\nabla u_\varepsilon^+ = 0$ on $\partial\Omega\times(0,T)$ and u_ε^+ satisfies the anisotropic Neumann boundary condition $a_p(x,\nabla u_\varepsilon^+)\cdot\nu = 0$ on $\partial\Omega\times(0,T)$. At last, we use the expansion

$$f(u_\varepsilon^+) = f(U_0) + qf'(U_0) + \frac{1}{2}q^2 f''(\theta)\,,$$

for some function $\theta(x,t)$ satisfying $U_0 < \theta < u_\varepsilon^+$.

Combining the above expressions with (1.6) and (3.5), we obtain $\mathcal{L}_0 u_\varepsilon^+ = E_1 + E_2 + E_3$, where

$$\begin{aligned}
E_1 &= -\frac{1}{\varepsilon^2}q\big(f'(U_0) + \frac{1}{2}qf''(\theta)\big) + {U_0}'p_t + q_t,\\
E_2 &= \frac{{U_0}''}{\varepsilon^2}\Big(1 - 2a(x,\nabla d_\phi)\Big),\\
E_3 &= \frac{{U_0}'}{\varepsilon}\Big((d_\phi)_t - \frac{1}{m(x)}\operatorname{div}\big[m(x)a_p(x,\nabla d_\phi)\big]\Big).
\end{aligned}$$

In order to estimate the above terms, we first present some useful inequalities. As $f'(0)$ and $f'(1)$ are strictly negative, we can find strictly positive constants b and m such that

$$\text{if} \quad U_0(z)\in[0,b]\cup[1-b,1] \qquad \text{then} \quad f'(U_0(z)) \le -m. \tag{6.9}$$

On the other hand, since the region $\{(x,z)\in\overline{\Omega}\times\mathbb{R}\,|\,U_0(z)\in[b,1-b]\}$ is compact and since ${U_0}'>0$ on $\mathbb{R}$, there exists a constant $a_1>0$ such that

$$\text{if} \quad U_0(z)\in[b,1-b] \quad \text{then} \quad {U_0}'(z) \ge a_1. \tag{6.10}$$

We now choose $M>0$ such that $|U_0|\le M-1$. We then define

$$F = \sup_{|z|\le M} |f(z)| + |f'(z)| + |f''(z)|\,, \tag{6.11}$$

$$\beta = \frac{m}{4}\,, \tag{6.12}$$

and choose σ that satisfies

$$0 < \sigma \leq \min(\sigma_0, \sigma_1, \sigma_2)\,, \tag{6.13}$$

where

$$\sigma_0 := \frac{a_1}{4\beta + F}, \quad \sigma_1 := \frac{1}{\beta+1}, \quad \sigma_2 := \frac{4\beta}{F(\beta+1)}.$$

Hence, combining (6.9) and (6.10), we obtain, using that $\sigma \leq \sigma_0$,

$$U_0{}'(z) - \sigma f'(U_0(z)) \geq 4\sigma\beta \qquad \text{for} \ \ z \in \mathbb{R}. \tag{6.14}$$

Now let $K > 1$ be arbitrary. In what follows we will show that $\mathcal{L}_0 u_\varepsilon^+ \geq 0$ provided that the constants ε_0 and L are appropriately chosen. From now on, we suppose that the following inequality is satisfied:

$$\varepsilon_0^2 L e^{LT} \leq 1\,. \tag{6.15}$$

Then, given any $\varepsilon \in (0, \varepsilon_0)$, since $\sigma \leq \sigma_1$, we have $0 \leq q(t) \leq 1$, hence

$$-M \leq u_\varepsilon^\pm(x,t) \leq M\,. \tag{6.16}$$

6.2.2 An estimate for E_1

A direct computation gives

$$E_1 = \frac{\beta}{\varepsilon^2}\, e^{-\beta t/\varepsilon^2}(I - \sigma\beta) + L e^{Lt}(I + \varepsilon^2 \sigma L),$$

where

$$I = U_0{}' - \sigma f'(U_0) - \frac{\sigma^2}{2} f''(\theta)(\beta e^{-\beta t/\varepsilon^2} + \varepsilon^2 L e^{Lt}).$$

In virtue of (6.14) and (6.16), we obtain

$$I \geq 4\sigma\beta - \frac{\sigma^2}{2} F(\beta + \varepsilon^2 L e^{LT}).$$

Then, in view of (6.15), using that $\sigma \leq \sigma_2$, we have $I \geq 2\sigma\beta$, which implies

$$E_1 \geq \frac{\sigma\beta^2}{\varepsilon^2} e^{-\beta t/\varepsilon^2} + 2\sigma\beta L e^{Lt} =: \frac{C_1}{\varepsilon^2} e^{-\beta t/\varepsilon^2} + C_1{}' L e^{Lt}.$$

6.2.3 An estimate for E_2

First, in the points where where $|d_\phi| < d_0$, by (6.4), we have $E_2 = 0$. Next we consider the points where $|d_\phi| \geq d_0$. We deduce from the definition of Λ_0 in (2.16) that

$$\begin{aligned} 0 \leq 2a(x, \nabla d_\phi(x,t)) &\leq (\Lambda^0)^2 |\nabla d_\phi(x,t)|^2 \\ &\leq (\Lambda^0)^2 \|\nabla d_\phi\|_\infty^2 := D < \infty. \end{aligned}$$

Applying Lemma 3.1 yields

$$|E_2| \leq \frac{C}{\varepsilon^2}(1+D)e^{-\lambda |d_\phi + \varepsilon p|/\varepsilon} \leq \frac{C}{\varepsilon^2}(1+D)e^{-\lambda(d_0/\varepsilon - |p|)}.$$

We remark that $0 < K - 1 \leq p \leq e^{LT} + K$, and suppose from now that the following assumption holds:

$$e^{LT} + K \leq \frac{d_0}{2\varepsilon_0}. \tag{6.17}$$

Then $\dfrac{d_0}{\varepsilon} - |p| \geq \dfrac{d_0}{2\varepsilon}$ so that, defining $C' := C(1+D)$,

$$|E_2| \leq \frac{C'}{\varepsilon^2} e^{-\lambda d_0/(2\varepsilon)} \leq C_2 := \frac{16C'}{(e\lambda d_0)^2}.$$

6.2.4 An estimate for E_3

We set

$$\mathcal{G}(x,t) = (d_\phi)_t(x,t) - \frac{1}{m(x)} \operatorname{div}\left[m(x) a_p(x, \nabla d_\phi(x,t))\right].$$

We recall that $d_\phi \in C^{3+\vartheta,(3+\vartheta)/2}$ in a neighborhood $\mathcal{V}$ of Γ, say

$$\mathcal{V} = \{(x,t) \in Q_T,\ |d_\phi(x,t)| < d_0\}.$$

Combining the fact that

$$2a(x, \nabla d_\phi(x,t)) = 1 \quad \text{ in } \mathcal{V},$$

with the definition of Λ^0 in (2.16), we see that

$$|\nabla d_\phi| \geq \frac{1}{\Lambda^0} \quad \text{ in } \mathcal{V}. \tag{6.18}$$

We also recall that $(x,p) \mapsto a(x,p)$ is of class $C^{3+\vartheta}_{loc}$ on $\overline{\Omega} \times \mathbb{R}^N \setminus \{0\}$. Since $|\nabla d_\phi|$ is bounded away from zero, it follows that $x \mapsto \mathcal{G}(x,t)$ is Lipschitz continuous on $\mathcal{V}$. By equation (6.6), we have that

$$\mathcal{G}(x,t) = 0 \quad \text{ on } \Gamma_t = \{x \in \Omega, d_\phi(x,t) = 0\},$$

and it follows from the mean value theorem applied separately on both sides of Γ_t that there exists a constant N_1 such that

$$|\mathcal{G}(x,t)| \leq N_1 |d_\phi(x,t)| \quad \text{for all } (x,t) \in \mathcal{V}. \tag{6.19}$$

Next, using Lemma 2.2, we remark that $\mathcal{G}$ is bounded on $\overline{\Omega} \times [0,T] \backslash \mathcal{V}$ so that there exists a constant N_2 such that

$$\sup_{\overline{\Omega} \times [0,T] \backslash \mathcal{V}} |\mathcal{G}(x,t)| \leq N_2. \tag{6.20}$$

By the inequalities (6.19) and (6.20), we deduce that

$$|\mathcal{G}(x,t)| \leq N |d_\phi(x,t)| \quad \text{in } Q_T,$$

with $N := \max(N_1, N_2/d_0)$. Applying Lemma 3.1 we deduce that

$$\begin{aligned} |E_3| \; &\leq NC \frac{|d_\phi|}{\varepsilon} e^{-\lambda |d_\phi/\varepsilon + p|} \\ &\leq NC \max_{y \in \mathbb{R}} |y| e^{-\lambda |y+p|} \\ &\leq NC \max(|p|, \frac{1}{\lambda}). \end{aligned}$$

Thus, recalling that $|p| \leq e^{Lt} + K$, we obtain

$$|E_3| \leq C_3 (e^{Lt} + K) + C_3{}',$$

where $C_3 := NC$ and $C_3{}' := NC/\lambda$.

6.2.5 Completion of the proof

Collecting the above estimates of E_1, E_2 and E_3 yields

$$\mathcal{L}_0 u_\varepsilon^+ \geq \frac{C_1}{\varepsilon^2} e^{-\beta t/\varepsilon^2} + (LC_1{}' - C_3) e^{Lt} - C_4,$$

where $C_4 := C_2 + KC_3 + C_3{}'$. Now, we set

$$L := \frac{1}{T} \ln \frac{d_0}{4\varepsilon_0}\,,$$

which, for ε_0 small enough, validates assumptions (6.15) and (6.17). If ε_0 is chosen sufficiently small (i.e. L sufficiently large), we obtain, for all $\varepsilon \in (0, \varepsilon_0)$,

$$\mathcal{L}_0 u_\varepsilon^+ \geq (LC_1{}' - C_3) e^{Lt} - C_4 \geq \frac{1}{2} LC_1{}' - C_4 \geq 0.$$

The proof of Lemma 6.1 is now completed. $\square$

7 Proof of Theorem 1.6 and Corollary 1.7

Let $\eta \in (0, \eta_0)$ be arbitrary. Choose β and σ that satisfy (6.12), (6.13) and

$$\sigma\beta \leq \frac{\eta}{3}. \tag{7.1}$$

By the generation of interface Theorem 5.1, there exist positive constants ε_0 and M_0 such that (5.2), (5.3) and (5.4) hold with the constant η replaced by $\sigma\beta/2$. Since, by the hypothesis (1.8) and the equality (2.21), $\nabla u_0(x) \cdot n_\phi(x) \neq 0$ everywhere on the initial interface $\Gamma_0 = \{x \in \Omega,\ u_0(x) = a\}$ and since Γ_0 is a compact hypersurface, we can find a positive constant M_1 such that

$$\begin{array}{llll} \text{if} & d_\phi(x,0) \geq \ \ M_1\varepsilon & \text{then} & u_0(x) \geq a + M_0\varepsilon, \\ \text{if} & d_\phi(x,0) \leq -M_1\varepsilon & \text{then} & u_0(x) \leq a - M_0\varepsilon. \end{array} \tag{7.2}$$

Now we define functions $H^+(x), H^-(x)$ by

$$H^+(x) = \begin{cases} 1 + \sigma\beta/2 & \text{if } d_\phi(x,0) > -M_1\varepsilon \\ \sigma\beta/2 & \text{if } d_\phi(x,0) \leq -M_1\varepsilon, \end{cases}$$

$$H^-(x) = \begin{cases} 1 - \sigma\beta/2 & \text{if } d_\phi(x,0) \geq \ M_1\varepsilon \\ -\sigma\beta/2 & \text{if } d_\phi(x,0) < \ M_1\varepsilon. \end{cases}$$

Then from the above observation we see that

$$H^-(x) \ \leq \ u^\varepsilon(x, \mu^{-1}\varepsilon^2|\ln\varepsilon|) \ \leq \ H^+(x), \tag{7.3}$$

for almost all $x \in \Omega$.

Next we fix a sufficiently large constant $K > 1$ such that

$$U_0(-M_1 + K) \geq 1 - \frac{\sigma\beta}{3} \quad \text{and} \quad U_0(M_1 - K) \leq \frac{\sigma\beta}{3}. \tag{7.4}$$

For this K, we choose ε_0 and L as in Lemma 6.1. We claim that

$$u_\varepsilon^-(x,0) \leq H^-(x), \qquad H^+(x) \leq u_\varepsilon^+(x,0), \tag{7.5}$$

for all $x \in \Omega$. We only prove the former inequality, as the proof of the latter is virtually the same. Then it amounts to showing that

$$u_\varepsilon^-(x,0) = U_0\big(\frac{d_0(x)}{\varepsilon} - K\big) - \sigma(\beta + \varepsilon^2 L) \ \leq \ H^-(x). \tag{7.6}$$

In the range where $d_\phi(x,0) < M_1\varepsilon$, the second inequality in (7.4) and the fact that U_0 is an increasing function imply

$$\begin{aligned} U_0\big(\frac{d_\phi(x,0)}{\varepsilon} - K\big) - \sigma(\beta + \varepsilon^2 L) &\leq U_0(M_1 - K) - \sigma\beta - \sigma\varepsilon^2 L \\ &\leq \frac{\sigma\beta}{3} - \sigma\beta \\ &\leq H^-(x). \end{aligned}$$

On the other hand, in the range where $d_\phi(x,0) \geq M_1\varepsilon$, we have

$$U_0\big(\frac{d_\phi(x,0)}{\varepsilon} - K\big) - \sigma(\beta + \varepsilon^2 L) \leq 1 - \sigma\beta$$
$$\leq H^-(x).$$

This proves (7.6), so that (7.5) is established.

Combining (7.3) and (7.5), we obtain

$$u_\varepsilon^-(x,0) \leq u^\varepsilon(x, \mu^{-1}\varepsilon^2|\ln\varepsilon|) \leq u_\varepsilon^+(x,0),$$

for almost all $x \in \Omega$. Since, by Lemma 6.1, u_ε^- and u_ε^+ are sub- and super-solutions of Problem (P^ε), the comparison principle yields

$$u_\varepsilon^-(x,t) \leq u^\varepsilon(x, t+t^\varepsilon) \leq u_\varepsilon^+(x,t), \tag{7.7}$$

for almost all $(x,t) \in Q_T$ that satisfies $0 \leq t \leq T - t^\varepsilon$, where we recall that $t^\varepsilon = \mu^{-1}\varepsilon^2|\ln\varepsilon|$. Note that, in view of (6.8), this is sufficient to prove Corollary 1.7. Now let C be a positive constant such that

$$U_0(C - e^{LT} - K) \geq 1 - \frac{\eta}{2} \quad \text{and} \quad U_0(-C + e^{LT} + K) \leq \frac{\eta}{2}. \tag{7.8}$$

One then easily checks, using (7.7) and (7.1), that, for ε_0 small enough and for almost all $(x,t) \in Q_T$, we have

$$\begin{array}{llll} \text{if} & d_\phi(x,t) \geq C\varepsilon & \text{then} & u^\varepsilon(x,t+t^\varepsilon) \geq 1-\eta, \\ \text{if} & d_\phi(x,t) \leq -C\varepsilon & \text{then} & u^\varepsilon(x,t+t^\varepsilon) \leq \eta, \end{array} \tag{7.9}$$

and

$$u^\varepsilon(x,t+t^\varepsilon) \in [-\eta, 1+\eta],$$

which completes the proof of Theorem 1.6. □

References

[1] M. Alfaro, D. Hilhorst and H. Matano, *The singular limit of the Allen-Cahn equation and the FitzHugh-Nagumo system*, J. Differential Equations **245** (2008), 505–565.

[2] J.W. Barrett, H. Garcke and R. Nürnberg, *Numerical approximation of anisotropic geometric evolution equations in the plane*, IMA J. Numer. Anal. **28** (2008), 292–330.

[3] J.W. Barrett, H. Garcke and R. Nürnberg, *A variational formulation of anisotropic geometric evolution equations in higher dimensions*, Numer. Math. **109** (2008), 1–44.

[4] G. Bellettini, P. Colli Franzone and M. Paolini, *Convergence of front propagation for anisotropic bistable reaction-diffusion equations*, Asymptot. Anal. **15** (1997), 325–358.

[5] G. Bellettini and M. Paolini, *Anisotropic motion by mean curvature in the context of Finsler geometry*, Hokkaido Math. J. **25** (1996), 537–566.

[6] G. Bellettini, M. Paolini and S. Venturini, *Some results on surface measures in calculus of variations*, Ann. Mat. Pura Appl. **170** (1996), 329–357.

[7] M. Beneš, *Mathematical analysis of phase-field equations with gradient coupling term*, In Partial Differential Equations– Theory and Numerical Solution (W. Jäger, J. Nečas, O. John, K. Najzar and J. Stará, eds.), pages 25–33, New York, 2000.

[8] M. Beneš, D. Hilhorst and R. Weidenfeld, *Interface dynamics for an anisotropic Allen-Cahn equation*, Nonlocal elliptic and parabolic problems, Banach Center Publ. 66, Polish Acad. Sci., Warsaw, 2004, 39–45.

[9] M. Beneš and K. Mikula, *Simulation of anisotropic motion by mean curvature—comparison of phase field and sharp interface approaches*, Acta Math. Univ. Comenian. **67** (1998), 17–42.

[10] G. Bouchitte, *Singular perturbations of variational problems arising from a two-phase transition model*, Appl. Math. Optim. **21** (1990), 289–315.

[11] L. Bronsard and R. V. Kohn, *Motion by mean curvature as the singular limit of Ginzburg–Landau dynamics*, J. Differential Equations **90** (1991), 211–237.

[12] X. Chen, *Generation and propagation of interfaces for reaction-diffusion equations*, J. Differential Equations **96** (1992), 116–141.

[13] C. M. Elliott and R. Schätzle, *The limit of the anisotropic double-obstacle Allen-Cahn equation*, Proc. Roy. Soc. Edinburgh A **126** (1996), 1217–1234.

[14] C. M. Elliott and R. Schätzle, *The limit of the fully anisotropic double-obstacle Allen-Cahn equation in the nonsmooth case*, SIAM. J. Math. Anal. **28** (1997), 274–303.

[15] H. Garcke, B. Nestler and B. Stoth, *Anisotropy in multiphase sytems: a phase-field approach*, Interfaces and Free Boundaries, **1** (1999), 175-198.

[16] Y. Giga, *Surface evolution equations*, Monographs in Mathematics 99, Birkhäuser Verlag, Basel, Boston, Berlin, 2006.

[17] Y. Giga and S. Goto, *Geometric evolution of phase-boundaries*, Mathematics and its applications 43, Springer-Verlag, Berlin, New York, 1992.

[18] A. Lunardi, *Analytic semigroups and optimal regularity in parabolic problems*, Progress in nonlinear differential equations and their applications 16, Birkhäuer Basel (1995).

[19] P. de Mottoni and M. Schatzman, *Development of interfaces in* $\mathbb{R}^n$, Proc. Royal Soc. Edinburgh **116A** (1990), 207–220.

[20] P. de Mottoni and M. Schatzman, *Geometrical evolution of developed interfaces*, Trans. Amer. Math. Soc. **347** (1995), 1533–1589.

[21] K.-I. Nakamura, H. Matano, D. Hilhorst and R. Schätzle, *Singular limit of a reaction-diffusion equation with a spatially inhomogeneous reaction term*, J. Stat. Phys. **95** (1999), 1165-1185.

[22] N. Owen, *Non convex variational problems with general singular perturbations*, Trans. Amer. Math. Soc. **310** (1988), 393–404.

[23] N. Owen and P. Sternberg, *Non convex variational problems with anisotropic perturbations*, Nonlinear Anal. **16** (1991), 705-719.

[24] M. Paolini, *Fattening in two dimensions obtained with a non-symmetric anisotropy: numerical simulations*, Acta Math. Univ. Comenian. (N.S.). **67** (1998), 43–55.